Some Modern Mathematics For Physicists and Other Outsiders

An Introduction to Algebra, Topology, and Functional Analysis

(Volume 2)

Paul Roman

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Preface

"Dammi conforto Dio et alegrança, e carità perfecta et amorança."
(Laudario 91 di Cortona, 13th Century.)

In these days of rapid proliferation of textbooks, any preface to a new book must begin with an apology, if not with a well-documented justification, concerning the raison d'être of this new volume. This is particularly true if the author ventures into a field which is not one of his primary competence. In the present case, my apology is quite simple. After many years of research in elementary particle theory and related topics, I realized that my "standard" mathematical background is inadequate to keep up with modern developments in theoretical physics. This realization was followed by a long period of hard study when I tried to dig out from the mountains of existing literature those concepts and tools without which I could no longer continue to be productive. I also realized that my graduate students should be spared this horrendous task and I introduced a new, one-year course on what, somewhat euphemistically, may be called "modern" mathematics. The outcome of these efforts is the present treatise.

It is obvious that a pragmatic survey of some relevant chapters of algebra, topology, measure theory, and functional analysis would not serve any useful purpose were it not unified by some central theme and presented in a manner that makes the hard work of absorbing the material not only useful but also truly enjoyable. In modern mathematics such a unifying viewpoint presents itself quite naturally: it is the pursuit of *structure*. Of course, a substantial segment of prospective learners will be impatient to gather, as fast as possible, readily available tools rather than to desire enchantment with beauty. Consequently, I tried hard to strike a healthy balance between structural investigations on one hand and practical theorems and methods on the other. I believe that this is one of the features which distinguishes the present volume from the host of other books, written both by professional specialists and by theoretical physicists. Another, and in my opinion, equally important feature is that I really started out from "scratch" and attempted to pave the way smoothly from elementary concepts to highly sophisticated and involved material. *The only prerequisite for the successful use of this book is a standard familiarity with basic calculus.* A superficial acquaintance with the elements of linear algebra (and perhaps with a few not quite elementary topics of classical analysis) will help, but is not essential.

Naturally, to some extent this book has the character of "selected topics." However, almost all topics covered in the earlier chapters *will be used* later on, and a strong thread of continuity ties the topics together. The attentive critique will observe that occasionally statements and examples are repeated, instead of referring the reader to earlier sections. This is done on purpose, so as to ease the student's work.

It was my firm policy always to proceed from the general to the specific. I took this decision not only so as to conform with the spirit of "modern mathematics," but also because I am convinced that, eventually, this approach is easier to digest and provides a much more stable and time-enduring knowledge than does the laborious method which starts with examples, proceeds to special cases and only then builds up the general theory.

On the other hand, I did not attempt to follow systematically the "Definition–Lemma–Theorem–Proof–Corollary–Remarks" sequence of most professional mathematics texts. Especially in the earlier chapters, many theorems are presented informally, "bringing out" the theorem by a series of observations rather than by first stating it and then supplying the detailed proof. Furthermore, many theorems (even some important ones) are stated without proof, in particular if the proof is atypical, lengthy and/or highly technical. From about Chapter 4 onward, both the rigor and the formal manner of presentation are increased, so that there is a certain amount of unevenness in style. This is the result of a conscious pedagogical decision, since I felt that the yet unexperienced reader should be at first spared the somewhat bleak succession of formal developments and should make fast progress in grasping basic structural features.

I believe that this book may fill the needs of most theoretical physicists (especially of those interested in quantum theory, high energy physics, relativity, modern statistical physics), many research engineers, and even other scientists who are concerned with structural problems, such as systems analysis. For students in these fields, this work is essentially a graduate level text. It is also possible that professional students of mathematics, in their earlier stage of studies (sophomore or in some cases junior undergraduate level), may profit from this volume inasmuch as it gives a survey of standard topics that they will have to study, eventually, in considerable depth.

This book may be used for self-study, since it is self-contained and, frankly speaking, originated from the author's self-study. For those who are already familiar with certain topics, this treatise may be used as a reference or quick refresher of once-learned but forgotten subjects. Pleasant experience has shown that the book will well serve as a textbook for a two- or three-semester course. Chapters 3 and 4 (with the essentials of the preceding two chapters) may be used for a short course or seminar in modern algebra. Similarly, Chapters 5 and 6 serve as a (somewhat superficial) guide to topology, and Chapters 7 and 8 to measure and integration. Those who wish to teach a leisurely one-semester course on the basics of functional analysis (especially Hilbert space theory) may start (for an already primed audience) with Chapter 9 and conclude with Chapter 12, with some material from Chapter 13 added on. Chapter 13, on the other hand, may be a good supplement for those who wish to penetrate deeper in the already absorbed field of Hilbert space operator theory. In connection with Chapters 12 and 13 I would like to mention that, having in mind the needs of the quantum theorists, I gave considerable attention to questions of domains and to unbounded operators, which is a topic unduly neglected in most introductory and even intermediate level texts.

In consequence of many factors, it was impossible to consistently show "applications" or even to indicate the areas where the discussed theorems and methods are particularly useful or necessary. I am aware that an occasional muttering about quantum theory does not indicate the physical importance of the discussed concept. However, this is a book on mathematics and I fervently hope that the student who patiently made his way through it will be able to understand any contemporary paper or book in the frontier areas of theoretical physics and to use with confidence any original and professional mathematical source that is needed to enlarge the knowledge he gained from this volume.

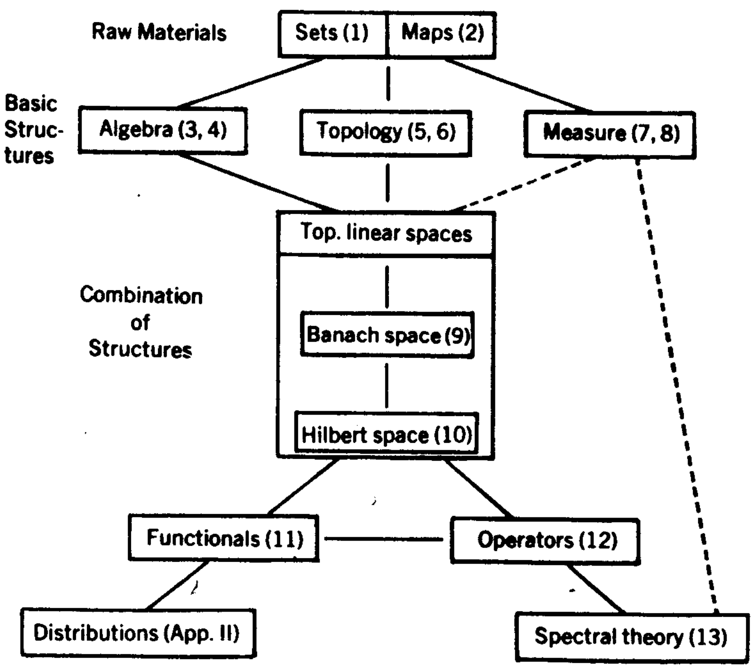
It was unavoidable that many topics, as important as those discussed in the text, had to be completely omitted from consideration. I particularly regret the omission of an introduction to topological (especially Lie) groups and their representation theory. Many topics of functional analysis that I discussed would have been well illustrated and put to direct use by the inclusion of a chapter on nonsingular Fredholm integral equations, to say the least. Commuting sets of operators and their spectral representation, complete sets of commuting operators, von Neumann algebras and related topics would have been also most desirable to be included, had space permitted this. I also regret the total omission of topics on differentiable structures. But I believe the interested reader is well equipped to study now these topics on his own. The annotated reading list in Appendix III may serve as a guide for him where to turn next.

PAUL ROMAN

Boston, Massachusetts

Organization of the Book

The logical connection of topics covered in this book is symbolized by the chart below. The numbers in the boxes refer to chapter numbers.



Each chapter consists of several sections and some sections have one or more subsections. Especially in the earlier parts of the book, the beginning paragraphs of a chapter or of a section may be quite long, without carrying any special number. (The Table of Contents will be helpful to locate subdivisions.) Subsection b of Section 4 in Chapter 13 is referred to by the symbol 13.4b, etc. Definitions and theorems are numbered according to their standing in a subsection or section. Thus, Theorem 13.4b(3) indicates the third theorem in Subsection 13.4b and Definition 13.4b(3) stands for the third definition in the same subsection. In the beginning part of a section (before a subsection, if any, is reached) the notation would be Theorem (or Definition) 11.2(3). Occasionally, one will find related theorems such as, say, Theorems 12.3b(3a) and 12.3b(3b) in succession. Figures and tables are numbered successively in each chapter. Thus, Fig. 13.3 refers to the third figure in the entire Chapter 13. Equations are numbered only if frequent reference is made to them or (especially in the beginning of the book) when special interest is drawn to them, for pedagogical reasons. Equations are numbered consecutively through each chapter. Thus, Eq. (13.4) refers to the fourth displayed and numbered equation in Chapter 13. Occasionally, in proofs, examples, or short discussions, for ease of reference during the discussion, equations are labeled by Greek letters. Thus, Eq. (γ) or simply (γ) refers to a so labeled, displayed equation in the immediate vicinity. There are many illustrative examples in the text. They are labeled in each subsection (or section), in alphabetic order, by Greek letters. Thus, we may refer to Example γ in Section 13.4 or Example α in Subsection 13.4a.

A special comment on problems is in order. Each subsection, most beginning parts of sections, and sometimes even the introductory parts of chapters are followed by a selection of problems. For example, Problem 13.4b–8 would refer to the eighth problem at the end of Subsection 13.4b, whereas Problem 12.2–3 is the third problem at the end of the initial part of Section 12.2 (just before Subsection 12.2a starts). Needless to say, the problems form an integral and even crucial part of the book. There are over 600 problems in toto and the reader is strongly urged to attempt all of them—but at the very least, half of them. (Experience shows, it can be done, within one academic year!) Some of the problems are in the same vein as the examples in the text, i.e. they are simple illustrations of theorems or definitions. Many other problems are designed to challenge the reader's understanding on the deeper level and to test his recall of earlier results. Finally, a certain proportion of problems is intended to extend the material that was covered in the text or to introduce some new concepts which, for one reason or another, could not be fitted smoothly into the main text. Occasionally, these problems introduce minor theorems which are then used in the main text, sometimes repeatedly. In many problems, and not only in the more difficult ones, hints are given to facilitate the solution. This is meant as an encouragement and the student is advised to first ignore the hints and find his own argument. For the successful tackling of the problems, no external reading whatsoever is necessary.

A final comment on references. In accord with standard custom in the mathematical textbook literature (and because of impractibility), no references to original publications are made. However, on occasions when we either omitted the proof of an important theorem or when details were neglected, we call the reader's attention to some standard text where he can look up the details. Such a reference will be indicated by a sentence like "See Helmberg[16], p. 215," which refers to Helmberg's book numbered as [16] in Appendix III of this treatise. Occasionally reference is made, quoting complete bibliographical data, to books that are not listed in Appendix III.

P. R.

Publisher's Note: For convenience to the student, this work has been divided into two volumes. Appendix II, which appears in Volume 2, does not appear in Volume 1 because it has no application there.

Acknowledgments

Departing from standard academic practice, I wish to use this occasion for expressing my gratitude, esteem, and indebtedness to several persons who, even though they were in no way directly connected with the project of this book, had a profound and formative influence on my mathematical education and on my attitude to mathematics.

First in chronological order, I wish to pay tribute to the memory of my Junior High School teacher, DR. J. MENDE. He perished tragically during the last days of the siege of Budapest, in January 1945. Next I recall the image of a friend of my early youth, the polyhistor P. T. MESSER. He was brutally murdered in the Spring of 1945, in Mauthausen, at the tender age of 21 years, before his talents became known to the world. It was only a few months later that my inspiring teacher, PROF. B. VON KERÉKJÁRTO, the well-known topologist of the 1930's, departed from this world, in consequence of illness and malnutrition suffered in the last phase of World War II. It was he who first revealed to me the fascinating interplay of analysis and algebra and, more generally, who made me aware of the unity of mathematics. Finally, even though I never had the opportunity to be his pupil or to work under his guidance, I wish to acknowledge the strong influence of Nobel Laureate PROF. E. P. WIGNER of Princeton University on my attitude to the role of mathematics in physics. I shall never forget the summer of 1946 when, amongst the ruins of my hometown, I first read a battered copy of his classic volume *Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren*, which was lent to me by my esteemed physics professor, the late K. F. NOVOBÁTZKY.

Turning to the conventional part of these acknowledgments, I would like to express my sincere thanks to a number of colleagues without whose interest, enthusiasm, and keen criticism this work would never have grown from a set of amateurish lecture notes into a (hopefully, more professional) book. To start with, I pay my tribute to MR. A. SIEGEL who, as a mathematics–physics junior at Boston University, attended my lectures in 1971–1972 and read, with unparalleled sharpness, the entire manuscript, discovering and correcting an alarming number of errors. Most helpful was also my graduate student and teaching assistant, MR. P. L. HUDDLESTON, who in the period 1970–1972 contributed a great deal to the eradication of mistakes and inconsistencies from the earlier version of the manuscript. Several other students who took the course, in particular MR. S. M. WOLFSON and MR. S. I. FINE, deserve most sincere thanks. Last but not least, I record my warmest gratitude to my friend, PROF. B. GRUBER of Southern Illinois University who carefully read the final manuscript and suggested numerous corrections and improvements.

P.R.

Contents of Volume 2

IIIB: Mapping of Topological Linear Spaces

APPENDICES

PART THREE

Combination of Systems: Functional Analysis

IIIA: Topological Linear Spaces

9

Banach Spaces

In Part Two of this book we have surveyed the basic structures of mathematics. In principle, the whole body of mathematics is not much more than a purposeful and systematic combination of the basic structures. Indeed, we learned "how to do algebra" (algebraic structures), how to formulate geometric problems and how to deal with convergence and continuity problems (topological systems), and we also saw how to handle the "rest" of geometry and calculus (measure theory and integration). It is worthwhile to recall that all these notions arose as a generalization of the properties of real numbers.

In summary, every given system may be characterized by specifying its algebraic, topological, and measure theoretic properties. For example, the real number system may be described as follows:†

(a) *Algebraically*: A commutative real division algebra of order one.
(b) *Topologically*: A locally compact, simply connected, locally connected, separable, complete metric space (actually, a Euclidean 1-space).
(c) *Measure theoretically*: A totally σ-finite complete measure space.

Everything there is to know about "one-dimensional" algebra, geometry, and calculus of functions on the reals, follows directly from these basic structures.

The attentive reader will notice that the three types of structure on the real line are not entirely independent of each other. In fact, there are significant interrelations among them and if one pays attention to these

†We assume that the usual, traditional structures are imposed on the underlying set.

relationships, a vast body of knowledge follows which could not be gleaned by concentrating on one or on another type of basic structure alone.

This simple observation suggests what actually the work of the modern mathematician is about. He spends his efforts on studying in detail and from specific viewpoints *the significant intertwining of the basic structures.* The combination of the basic mathematical structures leads to an endless stream of more and more exciting systems and to a practically unlimited wealth of discoveries.

There are several fundamental approaches possible for combining different structures. In this book, we shall concentrate on the process when *a topological structure is superimposed on a given algebraic structure.*† In this way we arrive at what may be called "topological algebra" but what, more commonly, is referred to as *functional analysis.* It should be mentioned that, historically speaking, the term "functional analysis" is sometimes used in a slightly different sense, to denote not so much a branch of mathematics, but rather a viewpoint, an approach with enormous unifying power. From this angle, functional analysis is characterized essentially as the study of mappings where both the domain and the codomain possess one or several interesting structures (algebraic, topological, and/or measure theoretic). With this viewpoint in mind, we may say that we already touched upon questions of this nature. The contraction mapping theorem, for example, is a good illustration. However, in our subsequent work we shall interpret the term "functional analysis" in the more precise sense as given above.

It may be thought that functional analysis should start with the superimposing of a topological structure on a given group structure. In this manner one is led to the concept of a topological group. But it turns out that these systems are quite complicated and it is better to start with endowing a *linear space* with a topological structure.‡ This leads to systems referred to as *topological linear spaces.* In this and in the next chapter we study the structure of such systems. Following our standard policy, we shall start with the most general type of a topological linear space and then, by imposing more and more special requirements, we shall proceed to the study of specific types.

†Measure theoretic concepts will also play an important role, primarily by constructing a topology via a metric which, in turn, is derived with the help of an integral. Integration theory will also appear as an indispensable tool in the detailed study of several concrete constructs.

‡Actually, the study of topological groups *uses*, as a most important tool, the theory of topological linear spaces.

9.1 GENERAL CONCEPTS CONCERNING TOPOLOGICAL LINEAR SPACES

Our aim is to combine a linear space with a superimposed topological structure. To ensure that the resulting combined structure be not entirely trivial, we will not use a completely arbitrary topology but rather one which is "compatible with the given linear space structure" and which is intimately interrelated with it. In this spirit, we start with the following fundamental

DEFINITION 9.1(1). *Let $\mathscr{L}$ be a linear space* (with elements denoted by $x, y, \ldots$). *Let K be the field of scalars*† *associated with $\mathscr{L}$* (with elements denoted by $\alpha, \beta, \ldots$). *Suppose that*

(i) *the vector sum $x + y$ is a continuous function on $\mathscr{L} \times \mathscr{L}$,*

(ii) *the scalar product αx is a continuous function on $K \times \mathscr{L}$.*

Then we say that $\mathscr{L}$ is a topological linear space.

Several comments are needed to clarify this definition. To start with, observe that a topological linear space is an ordered triple $(X, \mathscr{L}, \tau)$ consisting of an underlying set X, a linear space structure $\mathscr{L}$ on X, and a topology τ on X. However, in conformity with standard usage and without the risk of causing confusion, we suppress to indicate explicitly the underlying set X and use the same symbol $\mathscr{L}$ for both the set and for its algebraic structure. Next, we recall that the vector sum is a map $\mathscr{L} \times \mathscr{L} \to \mathscr{L}$ given by $(x, y) \mapsto x + y$. Now, we assumed that $\mathscr{L}$ (or more precisely, its underlying set X) has a topology τ. Then we tacitly stipulate that $\mathscr{L} \times \mathscr{L}$ (more precisely: $X \times X$) is endowed with the product topology $\tau \times \tau$ (cf. Section 5.2). Therefore, it is now *meaningful* to talk about the continuity of the vector sum map $\mathscr{L} \times \mathscr{L} \to \mathscr{L}$. Similarly recall that the scalar product is a map $K \times \mathscr{L} \to \mathscr{L}$ given by $(\alpha, x) \mapsto \alpha x$. We tacitly assume that the set of scalars (which, as noted in the footnote on this page, we take to be either the reals or the complex numbers) is endowed with its usual (metric) topology τ_K, and we also consider $K \times \mathscr{L}$ (more precisely, $K \times X$) endowed with the product topology $\tau_K \times \tau$. Thus, it is *meaningful* to talk about the continuity of the scalar product map $K \times \mathscr{L} \to \mathscr{L}$. The definition of a topological linear space then demands that $x + y$ and αx be continuous with respect to the relevant topologies of the respective domains and codomains.

†As we did in pure algebra, we restrict ourselves to the case when K is either the field of real numbers or that of the complex numbers. Correspondingly we speak of a real or complex topological linear space.