GILBERT G. WALTER
XIAOPING SHEN

WAVELETS and OTHER ORTHOGONAL SYSTEMS

SECOND EDITION

STUDIES IN ADVANCED MATHEMATICS
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GILBERT G. WALTER
Department of Mathematical Sciences
The University of Wisconsin–Milwaukee
Milwaukee, Wisconsin

XIAOPING SHEN
Department of Mathematics and Computer Sciences
Eastern Connecticut State University
Willimantic, Connecticut

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Preface to first edition

The subject of wavelets has evolved very rapidly in the last five or six years—so rapidly that many articles and books are already obsolete. However, there is one portion of wavelet theory that has reached a plateau, that is, the subject of orthogonal wavelets. The major concepts have become standard, and further development will probably be at the margins. In one sense they are no different than other orthogonal systems. They enable one to represent a function by a series of orthogonal functions. But there are notable differences: wavelet series converge pointwise when others don't, wavelet series are more localized and pick up edge effects better, wavelets use fewer coefficients to represent certain signals and images.

Unfortunately, not all is rosy. Wavelet expansions change excessively under arbitrary translations—much worse than Fourier series. The same is true for other operators such as convolution and differentiation.

In this book wavelets are presented in the same setting as other orthogonal systems, in particular Fourier series and orthogonal polynomials. Thus their advantages and disadvantages can be seen more directly.

The level of the book is such that it should be accessible to engineering and mathematics graduate students. It will for the most part assume a knowledge of analysis at the level of beginning graduate real and complex analysis courses. However, some of the later chapters are more technical and will require a stronger background. The Lebesgue integral will be used throughout. This has no practical effect on the calculation of integrals but does have a number of theoretical advantages.

Wavelets constitute the latest addition to the subject of orthogonal series, which are motivated by their usefulness in applications. In fact, orthogonal series have been associated with applications from their inception. Fourier invented trigonometric Fourier series in order to solve the partial differential equation associated with heat conduction and wave propagation. Other orthogonal series involving polynomials appeared in the 19th century. These too were closely related to problems in partial differential equations. The Legendre polynomials are used to find solutions to Laplace's equation in the sphere, the Hermite polynomials and the Laguerre polynomial for special cases of Schrödinger wave equations. These, together with Bessel functions, are special cases of Sturm-Liouville problems, which lead to orthogonal series, which are used to solve various partial differential equations.

The arrival of the Lebesgue integral in the early 20th century allowed the development of a general theory of orthogonal systems. While not oriented to applications, it allowed the introduction of new systems such as the Haar and Walsh systems, which have proven useful in signal processing. Also useful in this subject are the sinc functions and their translates, which form an orthogonal basis of a Paley-Wiener space. These are related to the prolate spheroidal functions, which are solutions both to an integral equation and a Sturm-Liouville problem.

The orthogonal sequences of wavelets, which are generalizations of the Haar system and the sinc system, have a number of unique properties. These make them useful in data compression, in image analysis, in signal processing, in numerical analysis, and in acoustics. They are particularly useful in digitizing data because of their decomposition and reconstruction algorithms. They also have better convergence properties than the classical orthogonal systems.

While the Lebesgue integral made a general theory of orthogonal systems possible, it is insufficiently general to handle many of the applications. In particular, the delta "function" or impulse function plays a central role in signal processing but is not a square integrable function. Fortunately a theory that incorporates such things appeared in the middle of the 20th century. This is the theory of "distributions," due mainly to L. Schwartz. It also is related to orthogonal systems in that it allows representation of distributions by orthogonal systems and also allows representations of functions by orthogonal distributions.

The body of the book is divided into 13 chapters of which the first 7 are expository and general while the remaining are more specialized and deal with applications to other areas. Each will be concerned with the use of or properties of orthogonal series.

In Chapter One we present two orthogonal systems that are prototypes for wavelets. These are the Haar system and the Shannon system, which have many, but not all, of the properties of orthogonal wavelets. They will be preceded by a section on general orthogonal systems. This is a standard theory that contains some results that will be useful in all of the particular examples.

Chapter Two will give a short introduction to tempered distributions. This is a relatively simple theory and is the only type of generalized function needed for much of orthogonal series. Many engineers still seem to apologize for their use of a "delta function". There is no need to do so since these are well defined proper mathematical entities. Included here also is the associated theory of Fourier transforms that enables one to take Fourier transforms of things like polynomial and trigonometric functions.

Chapter Three contains an introduction to the general theory of orthogonal wavelets. Their construction by a number of different schemes is given as are a number of their properties. These include their multiresolution property in which the terms of the series are naturally grouped at each resolution. The decomposition and reconstruction algorithms of Mallat, which give the coefficients at one resolution in terms of others, are presented here. Some of these properties are extended to tempered distributions in Chapter Five.

In Chapter Four we return to trigonometric Fourier series and discuss more detailed properties such as pointwise convergence and summability. These are fairly well known and many more details may be found in Zygmund's book. A short presentation on expansion of distributions in Fourier series is also presented.

In Chapter Five we also consider orthogonal systems in Sobolev spaces. These can be composed of delta functions as well as ordinary functions. In the former case we obtain an orthonormal series of delta function wavelets.

Chapter Six is devoted to another large class of examples, the orthogonal polynomials. The classical examples are defined and certain of their properties discussed. The Hermite polynomials are naturally associated with tempered distribution; properties of this connection are covered. Other orthogonal series are discussed in Chapter Seven.

Various kinds of convergence of orthogonal series are discussed in Chapter Eight. In particular, pointwise convergence of wavelet series is compared to that of other orthogonal systems. Also, the rate of convergence in Sobolev spaces is determined. Gibbs' phenomenon for wavelet series is compared to that for other series.

Chapter Nine deals with sampling theorems. These arise from many orthogonal systems including the trigonometric and polynomial systems. But the classical Shannon sampling theorem deals with wavelet subspaces for the Shannon wavelet. This can be extended to other wavelet subspaces as well. Both regular and irregular sampling points are considered.

In Chapter Ten we cover the relation between the translation operator and orthogonal systems. Wavelet expansions are not very well behaved with respect to this operator except for certain examples.

Chapter Eleven deals with analytic representation based on both Fourier series and wavelet. These are used to solve boundary value problems for harmonic functions in a half-plane with specified values on the real line.

Chapter Twelve covers probability density estimation with various orthogonal systems. Both Fourier series and Hermite series have been used, but wavelets come out the best.

Finally in the last chapter we cover the Karhunen-Loève theory for representing stochastic processes in terms of orthogonal series. An alternate formulation based on wavelets is developed.

Some of this text material was presented to a graduate course of mixed mathematics and engineering students. While not directly written as a text, it can serve as the basis for a modern course in Special Functions or in mathematics of signal processing. Problems are included at the end of each chapter. For the most part these are designed to aid in the understanding of the text material.

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G. G. Walter

Preface to second edition

In the years since the first edition of this book appeared, the subject of wavelets has continued its phenomenal growth. Much of this growth has been associated with new applications arising out of the multiscale properties of wavelets. Another source has been the widespread use of threshold methods to reduce the data requirements as well as the noise in certain signals. But in the area of wavelets as orthogonal systems, which is the main theme of this book, the growth has not been as marked. The principal new material has been in the area of multiwavelets, which, however, have not found their way into as many applications as the original theory. In addition, there seems to a resurgence of interest in nontensor product higher dimensional wavelets, but this area still needs some time to sort itself out.

In this new edition we have tried to correct many of the misprints and errors in the first edition (and in the process, have probably introduced others). We have reviewed the problems and introduced others in an effort to make their solution possible for average graduate students. We have also introduced a number of illustrations in an attempt to further clarify some of the concepts and examples. The first and fourth chapters remain approximately the same in this edition. The second chapter on distribution theory has been rewritten in order to make it somewhat more readable and self contained. Chapter three on orthogonal wavelet theory has been expanded with some additional examples: the raised cosine wavelets in closed form, and other Daubechies wavelets and their derivation. In Chapter five on wavelets and distributions, a section on impulse trains has been added. Chapter six on orthogonal polynomials remains essentially the same, while in Chapter seven a new section on an alternate approach to periodic wavelets has been added. In Chapter eight on pointwise convergence, an additional section on positive wavelets and their use in avoiding Gibbs' phenomenon is new. Chapter nine has been extensively revised and, in fact, has been split into two chapters, one devoted primarily to the Shannon sampling theorem and its properties and the new Chapter ten which concentrates more on sampling in other wavelet subspaces. New topics include irregular sampling in wavelet subspaces, hybrid wavelet sampling, Gibbs' phenomenon for sampling series in wavelet subspaces, and interpolating multiwavelets.

Chapter eleven on translation and dilation has only minor changes as does most of Chapter twelve except for a few pages on wavelets of entire analytic functions. In Chapter thirteen on statistics a number of new topics have been added. These include positive wavelet density estimators, density estimators with noisy data, and threshold methods. Some additional calculations involving some of these estimators are also included. Chapter fourteen, which deals with stochastic processes, has some new material on cyclostationary processes.

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The contributions of many individuals appear in this new edition. In particular the authors wish to acknowledge the work of Youming Liu, Hong-tae Shim, and Luchuan Cai which is covered in more detail here.

Gilbert G. Walter and Xiaoping Shen

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