SHAFAREVICH

BASIC ALGEBRAIC GEOMETRY

Varieties in Projective Space

Second Edition

基础代数几何 第1卷

第2版

Springer-Verlag

世界图书出版公司

Igor R. Shafarevich

Basic Algebraic Geometry 1

Second, Revised and Expanded Edition

Springer-Verlag
世界图书出版公司
北京·广州·上海·西安

Igor R. Shafarevich
Steklov Mathematical Institute
Ul. Vavilova 42, 117966 Moscow, Russia

Translator:
Miles Reid
Mathematics Institute, University of Warwick
Coventry CV4 7AL, England
e-mail: Miles@Maths.Warwick.Ac.UK.

With 21 Figures

The title of the original Russian edition:
Osnovy algebraicheskoj geometrii, tom 1

ISBN 3-540-54812-2 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-54812-2 Springer-Verlag New York Berlin Heidelberg

ISBN 3-540-08264-6 1. Auflage Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-08264-6 1st edition Springer-Verlag New York Berlin Heidelberg

Library of Congress Cataloging-in-Publication Data
Shafarevich, I. R. (Igor' Rostislavovich), 1923- [Osnovy algebraicheskoi geometrii. English] Basic algebraic geometry/Igor R. Shafarevich; [translator, Miles Reid]. - 2nd rev. and expanded ed. p. cm. "Springer study edition." Includes bibliographical references and indexes. ISBN 3-540-54812-2 (Berlin: acid-free: v. 1) - ISBN 0-387-54812-2 (New York: acid-free: v. 1). - ISBN 3-540-57554-5 (Berlin: acid-free: v. 2). - ISBN 0-387-57554-5 (New York: acid-free: v. 2). 1. Geometry, Algebraic. I. Title. QA564.S4513 1994 516.3'5. - dc20

Reprinted in China by Beijing World Publishing Corporation, 1998

Preface

The first edition of this book came out just as the apparatus of algebraic geometry was reaching a stage that permitted a lucid and concise account of the foundations of the subject. The author was no longer forced into the painful choice between sacrificing rigour of exposition or overloading the clear geometrical picture with cumbersome algebraic apparatus.

The 15 years that have elapsed since the first edition have seen the appearance of many beautiful books treating various branches of algebraic geometry. However, as far as I know, no other author has been attracted to the aim which this book set itself: to give an overall view of the many varied aspects of algebraic geometry, without going too far afield into the different theories. There is thus scope for a second edition. In preparing this, I have included some additional material, rather varied in nature, and have made some small cuts, but the general character of the book remains unchanged.

The three parts of the book now appear as two separate volumes. Book 1 corresponds to Part I, Chapters I–IV, of the first edition. Here quite a lot of material of a rather concrete geometric nature has been added: the first section, forming a bridge between coordinate geometry and the theory of algebraic curves in the plane, has been substantially expanded. More space has been given over to concrete algebraic varieties: Grassmannian varieties, plane cubic curves and the cubic surface. The main role that singularities played in the first edition was in giving rigorous definition to situations we wished to avoid. The present edition treats a number of questions related to degenerate fibres in families: degenerations of quadrics and of elliptic curves, the Bertini theorems. We discuss the notion of infinitely near points of algebraic curves on surfaces and normal surface singularities. Finally, some applications to number theory have been added: the zeta function of algebraic varieties over a finite field and the analogue of the Riemann hypothesis for elliptic curves.

Books 2 and 3 corresponds to Parts II and III, Chapters V–IX of the first edition. They treat the foundations of the theory of schemes, abstract algebraic varieties and algebraic manifolds over the complex number field. As in the Book 1 there are a number of additions to the text. Of these, the following are the two most important. The first is a discussion of the notion of moduli spaces, that is, algebraic varieties that classify algebraic or geometric objects of some type; as an example we work out the theory of the Hilbert polynomial and the Hilbert scheme. I am very grateful to V. I. Danilov for a series of recommendations on this subject. In particular the proof of Chap. VI, 4.3, Theorem 3 is due to him. The second addition is the definition and basic properties of a Kähler metric and a description (without proof) of Hodge's theorem.

For the most part, this material is taken from my old lectures and seminars, from notes provided by members of the audience. A number of improvements of proofs have been borrowed from the books of Mumford and Fulton. A whole series of misprints and inaccuracies in the first edition were pointed out by readers, and by readers of the English translation. Especially valuable was the advice of Andrei Tyurin and Viktor Kulikov; in particular, the proof of Chap. IV, 4.3, Theorem 3 was provided by Kulikov. I offer sincere thanks to all these.

Many substantial improvements are due to V. L. Popov, who edited the second edition, and I am very grateful to him for all the work and thought he has put into the book. I have the pleasure, not for the first time, of expressing my deep gratitude to the translator of this book, Miles Reid. His thoughtful work has made it possible to patch up many uneven places and inaccuracies, and to correct a few mathematical errors.

Prerequisites

The nature of the book requires the algebraic apparatus to be kept to a minimum. In addition to an undergraduate algebra course, we assume known basic material from field theory: finite and transcendental extensions (but not Galois theory), and from ring theory: ideals and quotient rings. In a number of isolated instances we refer to the literature on algebra; these references are chosen so that the reader can understand the relevant point, independently of the preceding parts of the book being referred to. Somewhat more specialised algebraic questions are collected together in the Algebraic Appendix at the end of Book 1. The preface to Books 2–3 contains recommended further reading in Algebraic Geometry.

Preface to the 1972 Edition

Algebraic geometry played a central role in 19th century math. The deepest results of Abel, Riemann, Weierstrass, and many of the most important works of Klein and Poincaré were part of this subject.

The turn of the 20th century saw a sharp change in attitude to algebraic geometry. In the 1910s Klein[1] writes as follows: "In my student days, under

[1] Klein, F.: Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert, Grundlehren Math. Wiss. 24, Springer-Verlag, Berlin 1926. Jrb. 52, 22, p. 312

the influence of the Jacobi tradition, Abelian functions were considered as the unarguable pinnacle of math. Every one of us felt the natural ambition to make some independent progress in this field. And now? The younger generation scarcely knows what Abelian functions are." (From the modern viewpoint, the theory of Abelian functions is an analytic aspect of the theory of Abelian varieties, that is, projective algebraic group varieties; compare the historical sketch.)

Algebraic geometry had become set in a way of thinking too far removed from the set-theoretic and axiomatic spirit that determined the development of math at the time. It was to take several decades, during which the theories of topological, differentiable and complex manifolds, of general fields, and of ideals in sufficiently general rings were developed, before it became possible to construct algebraic geometry on the basis of the principles of set-theoretic math.

Towards the middle of the 20th century algebraic geometry had to a large extent been through such a reconstruction. Because of this, it could again claim the place it had once occupied in math. The domain of application of its ideas had grown tremendously, both in the direction of algebraic varieties over arbitrary fields and of more general complex manifolds. Many of the best achievements of algebraic geometry could be cleared of the accusation of incomprehensibility or lack of rigour.

The foundation for this reconstruction was algebra. In its first versions, the use of precise algebraic apparatus often led to a loss of the brilliant geometric style characteristic of the preceding period. However, the 1950s and 60s have brought substantial simplifications to the foundation of algebraic geometry, which have allowed us to come significantly closer to the ideal combination of logical transparency and geometric intuition.

The purpose of this book is to treat the foundations of algebraic geometry across a fairly wide front, giving an overall account of the subject, and preparing the ground for a study of the more specialised literature. No prior knowledge of algebraic geometry is assumed on the part of the reader, neither general theorems, nor concrete examples. Therefore along with development of the general theory, a lot of space is devoted to applications and particular cases, intended to motivate new ideas or new ways of formulating questions.

It seems to me that, in the spirit of the biogenetic law, the student who repeats in miniature the evolution of algebraic geometry will grasp the logic of the subject more clearly. Thus, for example, the first section is concerned with very simple properties of algebraic plane curves. Similarly, Part I of the book considers only algebraic varieties in an ambient projective space, and the reader only meets schemes and the general notion of a variety in Part II.

Part III treats algebraic varieties over the complex number field, and their relation to complex analytic manifolds. This section assumes some acquaintance with basic topology and the theory of analytic functions.

I am extremely grateful to everyone whose advice helped me with this book. It is based on lecture notes from several courses I gave in Moscow University. Many participants in the lectures or readers of the notes have provided me with useful remarks. I am especially indebted to the editor B. G. Moishezon for a large number of discussions which were very useful to me. A series of proofs contained in the book are based on his advice.

Translator's Note

Shafarevich's book is the fruit of lecture courses at Moscow State University in the 1960s and early 1970s. The style of Russian mathematical writing of the period is very much in evidence. The book does not aim to cover a huge volume of material in the maximal generality and rigour, but gives instead a well-considered choice of topics, with a human-oriented discussion of the motivation and the ideas, and some sample results (including a good number of hard theorems with complete proofs). In view of the difficulty of keeping up with developments in algebraic geometry during the 1960s, and the extraordinary difficulties faced by Soviet mathematicians of that period, the book is a tremendous achievement.

The student who wants to get through the technical material of algebraic geometry quickly and at full strength should perhaps turn to Hartshorne's book [35]; however, my experience is that some graduate students (by no means all) can work hard for a year or two on Chaps. II–III of Hartshorne, and still know more-or-less nothing at the end of it. For many students, it's just not feasible both to do the research for a Ph. D. thesis and to master all the technical foundations of algebraic geometry at the same time. In any case, even if you have mastered everything in scheme theory, your research may well take you into number theory or differential geometry or representation theory or math physics, and you'll have just as many new technical things to learn there. For all such students, and for the many specialists in other branches of math who need a liberal education in algebraic geometry, Shafarevich's book is a must.

The previous English translation by the late Prof. Kurt Hirsch has been used with great profit by many students over the last two decades. In preparing the new translation of the revised edition, in addition to correcting a few typographical errors and putting the references into English alphabetical order, I have attempted to put Shafarevich's text into the language used by the present generation of English-speaking algebraic geometers. I have in a few cases corrected the Russian text, or even made some fairly arbitrary changes when the original was already perfectly all right, in most case with the author's explicit or implicit approval. The footnotes are all mine: they are mainly pedantic in nature, either concerned with minor points of terminology, or giving references for proofs not found in the main text; my references do not necessarily follow Shafarevich's ground-rule of being a few pages accessible to the general reader, without obliging him or her to read a whole book, and so may not be very useful to the beginning graduate student. It's actually quite demoralising to realise just how difficult or obscure the literature can be on some of these points, at the same time as many of the easier points are covered in any number of textbooks. For example: (1) the "principle of conservation of number" (algebraic equivalence implies numerical equivalence); (2) the Néron–Severi theorem (stated as Chap. 3, 4.4, Theorem D); (3) a punctured neighbourhood of a singular point of a normal variety over $\mathbb{C}$ is connected; (4) Chevalley's theorem that every algebraic group is an extension of an Abelian variety by an affine (linear) group. A practical solution for the reader is to take the statements on trust for the time being.

The two volumes have a common index and list of references, but only the second volume has the references for the historical sketch.

Table of Contents Volume 1

BOOK 1. Varieties in Projective Space

Table of Contents Volume 2

BOOK 2. Schemes and Varieties

BOOK 3. Complex Algebraic Varieties and Complex Manifolds

Chapter I. Basic Notions

1. Algebraic Curves in the Plane

Chapter I discusses a number of the basic ideas of algebraic geometry; this first section treats some examples to prepare the ground for these ideas.

1.1. Plane Curves

An *algebraic plane curve* is a curve consisting of the points of the plane whose coordinates x, y satisfy an equation

$$f(x,y) = 0, \tag{1}$$

where f is a nonconstant polynomial. Here we fix a field k and assume that the coordinates x, y of points and the coefficients of f are elements of k. We write $\mathbf{A}^2$ for the *affine plane*, the set of points (a, b) with a, $b \in k$; because the affine plane $\mathbf{A}^2$ is not the only ambient space in which algebraic curves will be considered – we will be meeting others presently – an algebraic curve as just defined is called an *affine* plane curve.

The degree of the equation (1), that is, the degree of the polynomial $f(x,y)$, is also called the *degree* of the curve. A curve of degree 2 is called a *conic*, and a curve of degree 3 a *cubic*.

It is well known that the polynomial ring $k[X,Y]$ is a unique factorisation domain (UFD), that is, any polynomial f has a unique factorisation $f = f_1^{k_1} \cdots f_r^{k_r}$ (up to constant multiples) as a product of irreducible factors f_i, where the irreducible f_i are nonproportional, that is, $f_i \neq \alpha f_j$ with $\alpha \in k$ if $i \neq j$. Then the algebraic curve X given by $f = 0$ is the union of the curves X_i given by $f_i = 0$. A curve is *irreducible* if its equation is an irreducible polynomial. The decomposition $X = X_1 \cup \cdots \cup X_r$ just obtained is called a decomposition of X into irreducible components.

In certain cases, the notions just introduced turn out not to be well defined, or to differ wildly from our intuition. This is due to the specific nature of the field k in which the coordinates of points of the curve are taken. For example if $k = \mathbf{R}$ then following the above terminology we should call the point $(0,0)$ a "curve", since it is defined by the equation $x^2 + y^2 = 0$. Moreover, this "curve" should have "degree" 2, but also any other even number,