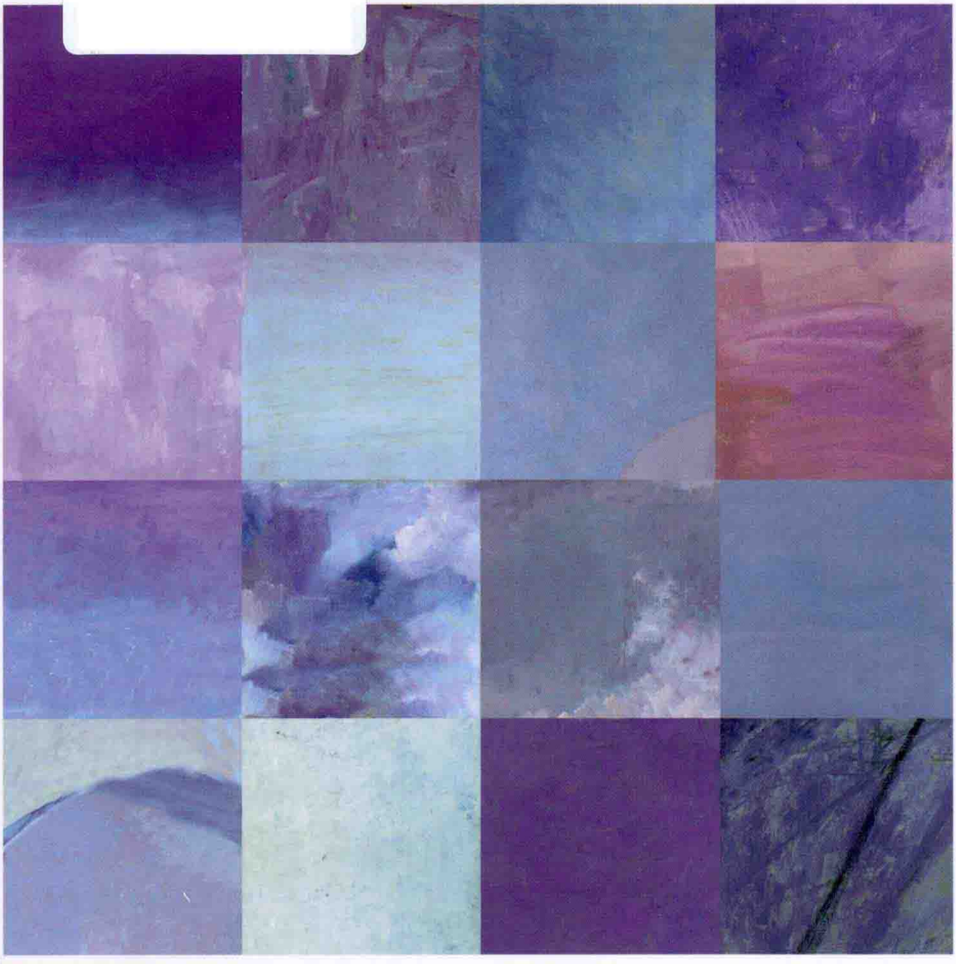
MATRICES

ALGEBRA, ANALYSIS AND APPLICATIONS



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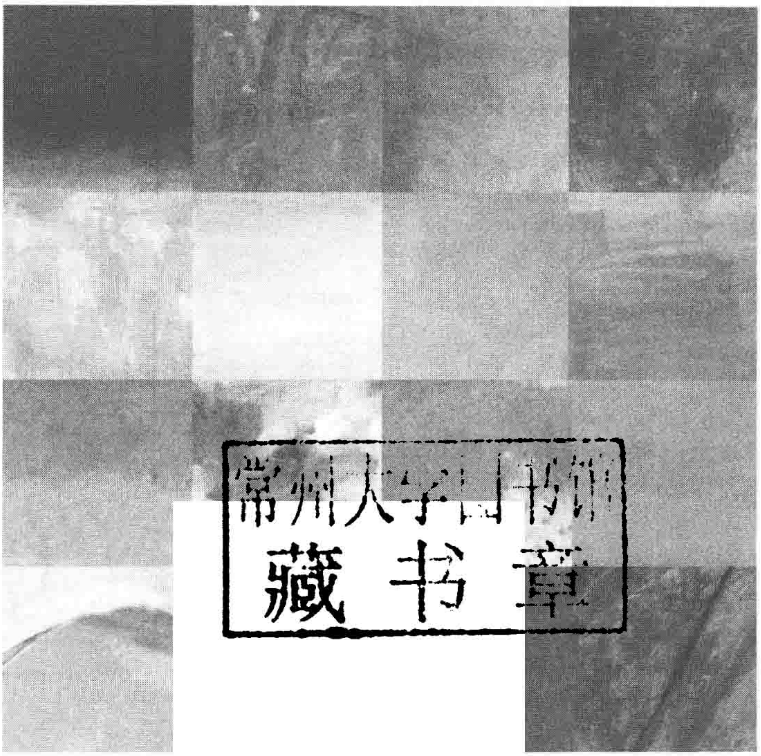

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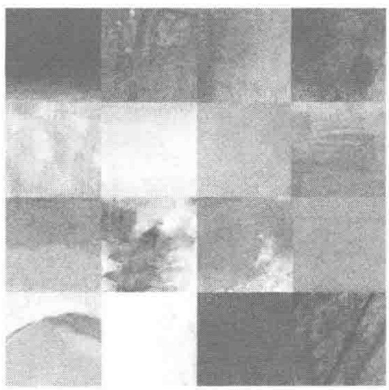

To the memory of my parents: Aron and Golda Friedland

Preface

Linear algebra and matrix theory are closely related subjects that are used extensively in pure and applied mathematics, bioinformatics, computer science, economy, engineering, physics and social sciences. Some results in these subjects are quite simple and some are very advanced and technical. This book reflects a very personal selection of the topics in matrix theory that the author was actively working on in the past 40 years. Some of the topics are very classical and available in a number of books. Other topics are not available in the books that are currently on the market. The author lectured several times certain parts of this book in graduate courses in Hebrew University of Jerusalem, University of Illinois at Chicago, Technion, and TU-Berlin.

The book consists of seven chapters which are somewhat interdependent. Chapter 1 discusses the fundamental notions of Linear Algebra over general and special integral domains. Chapter 2 deals with well-known canonical form: Jordan canonical form, Kronecker canonical form, and their applications. Chapter 3 discusses functions of matrices and analytic similarity with respect to one complex variable. Chapter 4 is devoted to linear operators over finite dimensional inner product spaces. Chapter 5 is a short chapter on elements of multilinear algebra. Chapter 6 deals with non-negative matrices. Chapter 7 discusses various topics as norms, complexity problem of the convex hull of a tensor product of certain two convex sets, variation of tensor power and spectra, inverse eigenvalue problems for non-negative matrices, and cones.

This book started as an MRC report "Spectral theory of matrices", 1980, University of Madison, Wisconsin. I continued to work on this book in Hebrew University of Jerusalem, University of Illinois at Chicago and in Technion and Technical University of Berlin during my Sabbaticals in 2000 and 2007–2008 respectively.

I thank Eleanor Smith for reading parts of this book and for her useful remarks.

Chicago, January 1, 2015

Contents

Chapter 1

Domains, Modules and Matrices

1.1 Rings, Domains and Fields

Definition 1.1.1 *A nonempty set* R *is called a ring if* R *has two binary operations, called addition and multiplication and denoted by* $a+b$ *and* ab *respectively, such that for all* $a, b, c \in \mathrm{R}$ *the following holds*:

$$a + b \in \mathrm{R}; \tag{1.1.1}$$

$$a + b = b + a \quad \text{(the commutative law)}; \tag{1.1.2}$$

$$(a + b) + c = a + (b + c) \quad \text{(the associative law)}; \tag{1.1.3}$$

$$\exists\, 0 \in \mathrm{R} \text{ such that } a + 0 = 0 + a = a, \ \forall\, a \in \mathrm{R}; \tag{1.1.4}$$

$$\forall\, a \in \mathrm{R},\ \exists - a \in \mathrm{R} \text{ such that } a + (-a) = 0; \tag{1.1.5}$$

$$ab \in \mathrm{R}; \tag{1.1.6}$$

$$a(bc) = (ab)c \quad \text{(the associative law)}; \tag{1.1.7}$$

$$a(b + c) = ab + ac, \quad (b + c)a = ba + ca, \quad \text{(the distributive laws)}. \tag{1.1.8}$$

R has an *identity* element 1 if $a1 = 1a$ for all $a \in \mathrm{R}$. R is called *commutative* if

$$ab = ba, \quad \text{for all } a, b \in \mathrm{R}. \tag{1.1.9}$$

Note that the properties (1.1.2)–(1.1.8) imply that $a0 = 0a = 0$. If a and b are two nonzero elements such that

$$ab = 0, \tag{1.1.10}$$

then a and b are called *zero divisors.*

Definition 1.1.2 $\mathbb{D}$ *is called an integral domain if* $\mathbb{D}$ *is a commutative ring without zero divisors which contains an identity element* 1.

The classical example of an integral domain is the ring of integers $\mathbb{Z}$. In this book we shall use the following example of an integral domain.

Example 1.1.3 *Let* $\Omega \subset \mathbb{C}^n$ *be a nonempty set. Then* $\mathrm{H}(\Omega)$ *denotes the ring of functions* $f(z_1, \ldots, z_n)$ *such that for each* $\zeta \in \Omega$ *there exists an open neighborhood* $O(f, \zeta)$ *of* ζ *on which* f *is analytic. If* Ω *is open we assume that* f *is defined only on* Ω. *If* Ω *consists of one point* ζ *then* H_ζ *stands for* $\mathrm{H}(\{\zeta\})$.

Recall that $\Omega \subset \mathbb{C}^n$ is called *connected*, if in the relative topology on Ω, induced by the standard topology on $\mathbb{C}^n$, the only subsets of Ω which are both open in Ω and closed in Ω are $\emptyset$ and Ω. Note that the zero element is the zero function of $\mathrm{H}(\Omega)$ and the identity element is the constant function which is equal to 1. The properties of analytic functions imply that $\mathrm{H}(\Omega)$ is an integral domain if and only if Ω is a *connected* set. In this book, we shall assume that Ω is connected unless otherwise stated. See [Rud74] and [GuR65] for properties of analytic functions in one and several complex variables.

Definition 1.1.4 *A nonempty* $\Omega \subset \mathbb{C}^n$ *is called a domain if* Ω *is an open connected set.*

For $a, b \in \mathbb{D}$, a *divides* b, (or a is a *divisor* of b), denoted by $a|b$, if $b = ab_1$ for some $b_1 \in \mathbb{D}$. An element a is called *invertible*, (unit, unimodular), if $a|1$. $a, b \in \mathbb{D}$ are *associates*, denoted by $a \equiv b$, if $a|b$ and $b|a$. Let $\{\{b\}\} = \{a \in \mathbb{D} :\ a \equiv b\}$. The associates of a and units are called *improper* divisors of a. For an invertible a denote by a^{-1} the unique element such that

$$aa^{-1} = a^{-1}a = 1. \tag{1.1.11}$$

$f \in \mathrm{H}(\Omega)$ is invertible if and only if f does not vanish at any point of Ω.

Definition 1.1.5 *A field $\mathbb{F}$ is an integral domain $\mathbb{D}$ such that any nonzero element is invertible. A field $\mathbb{F}$ has characteristic 0 if for any nonzero integer n and a nonzero element $f \in \mathbb{F}$ $nf \neq 0$.*

The familiar examples of fields are the set of rational numbers $\mathbb{Q}$, the set of real numbers $\mathbb{R}$, and the set of complex numbers $\mathbb{C}$. Note that the characteristic of $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ is 0. Given an integral domain $\mathbb{D}$ there is a standard way to construct the *field* $\mathbb{F}$ of its *quotients*. $\mathbb{F}$ is formed by the set of equivalence classes of all quotients $\frac{a}{b}, b \neq 0$ such that

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}, \quad \frac{a}{b}\frac{c}{d} = \frac{ac}{bd}, \quad b, d \neq 0. \tag{1.1.12}$$

Definition 1.1.6 *For $\Omega \subset \mathbb{C}^n, \zeta \in \mathbb{C}^n$ let $\mathcal{M}(\Omega), \mathcal{M}_\zeta$ denote the quotient fields of $\mathrm{H}(\Omega), \mathrm{H}_\zeta$ respectively.*

Definition 1.1.7 *Let $\mathbb{D}[x_1, \ldots, x_n]$ be the ring of all polynomials in n variables with coefficients in $\mathbb{D}$:*

$$p(x_1, \ldots, x_n) = \sum_{|\alpha| \le m} a_\alpha x^\alpha, \text{ for some } m \in \mathbb{N}, \tag{1.1.13}$$

$$\text{where } \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n, \quad |\alpha| := \sum_{i=1}^n \alpha_i, \quad x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

Sometimes we denote $\mathbb{D}[x_1, \ldots, x_n]$ and $p(x_1, \ldots, x_n)$ by $\mathbb{D}[\mathbf{x}]$ and $p(\mathbf{x})$ respectively.

The *degree* of $p(x_1, \ldots, x_n) \neq 0$ (denoted deg p) is the largest natural number d such that there exists $a_\alpha \neq 0$ with $|\alpha| = d$. (deg $0 = -\infty$.) A polynomial p is called *homogeneous* if $a_\alpha = 0$ for all $|\alpha| <$ deg p. It is a standard fact that $\mathbb{D}[x_1, \ldots, x_n]$ is an integral domain. (See Problems 2–3 below.) As usual $\mathbb{F}(x_1, \ldots, x_n)$ denotes the quotient field of $\mathbb{F}[x_1, \ldots, x_n]$.

Problems

1. Let $C[a, b]$ be the set of real valued continuous functions on the interval $[a, b], a < b$. Show that $C[a, b]$ is a commutative ring with identity and zero divisors.

2. Let $\mathbb{D}$ be an integral domain. Prove that $\mathbb{D}[x]$ is an integral domain.

3. Prove that $\mathbb{D}[x_1, \ldots, x_n]$ is an integral domain. (Use the previous problem and the identity $\mathbb{D}[x_1, \ldots, x_n] = \mathbb{D}[x_1, \ldots, x_{n-1}][x_n]$.)

4. Let $p(x_1, \ldots, x_n) \in \mathbb{D}[x_1, \ldots, x_n]$. Show that $p = \sum_{i \le \deg p} p_i$, where each p_i is either a zero polynomial or a homogeneous polynomial of degree i for $i \geq 0$. If p is not a constant polynomial then $m = \deg p \geq 1$ and $p_m \neq 0$. The polynomial p_m is called the *principal part* of p and is denoted by p_π. (If p is a constant polynomial then $p_\pi = p$.)

5. Let $p, q \in \mathbb{D}[x_1, \ldots, x_n]$. Show $(pq)_\pi = p_\pi q_\pi$.

6. Let $\mathbb{F}$ be a field with two elements at least which does not have characteristic 0. Show that there exists a unique prime integer $p \geq 2$ such that $pf = 0$ for each $f \in \mathbb{F}$. p is called the characteristic of $\mathbb{F}$.

1.2 Bezout Domains

Let $a_1, \ldots, a_n \in \mathbb{D}$. Assume first that not all of $a_1, \ldots, a_n$ are equal to zero. An element $d \in \mathbb{D}$ is a *greatest common divisor* (g.c.d.) of $a_1, \ldots, a_n$ if $d | a_i$ for $i = 1, \ldots, n$, and for any d' such that $d' | a_i, i = 1, \ldots, n$, $d' | d$. Denote by $(a_1, \ldots, a_n)$ any g.c.d. of $a_1, \ldots, a_n$. Then $\{\{(a_1, \ldots, a_n)\}\}$ is the equivalence class of all g.c.d. of $a_1, \ldots, a_n$.