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An Introduction to the Mathematical Theory of Inverse Problems

逆问题数学理论导论

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Preface

Following Keller [119] we call two problems *inverse* to each other if the formulation of each of them requires full or partial knowledge of the other. By this definition, it is obviously arbitrary which of the two problems we call the direct and which we call the inverse problem. But usually, one of the problems has been studied earlier and, perhaps, in more detail. This one is usually called the *direct* problem, whereas the other is the *inverse* problem. However, there is often another, more important difference between these two problems. Hadamard (see [91]) introduced the concept of a *well-posed problem*, originating from the philosophy that the mathematical model of a physical problem has to have the properties of uniqueness, existence, and stability of the solution. If one of the properties fails to hold, he called the problem *ill-posed*. It turns out that many interesting and important inverse problems in science lead to ill-posed problems, while the corresponding direct problems are well-posed. Often, existence and uniqueness can be forced by enlarging or reducing the solution space (the space of "models"). For restoring stability, however, one has to change the topology of the spaces, which is in many cases impossible because of the presence of measurement errors. At first glance, it seems to be impossible to compute the solution of a problem numerically if the solution of the problem does not depend continuously on the data, i.e., for the case of ill-posed problems. Under additional a priori information about the solution, such as smoothness and bounds on the derivatives, however, it is possible to restore stability and construct efficient numerical algorithms.

We make no claim to cover all of the topics in the theory of inverse problems. Indeed, with the rapid growth of this field and its relationship to many fields of natural and technical sciences, such a task would certainly be impossible for a single author in a single volume. The aim of this book is twofold: First, we will introduce the reader to the basic notions and difficulties encountered with ill-posed problems. We will then study the basic properties of regularization methods for *linear* ill-posed problems. These methods can roughly be classified into two groups, namely, whether the regularization parameter is chosen a priori or a posteriori. We will study some of the most important regularization schemes in detail.

The second aim of this book is to give a first insight into two special *nonlinear* inverse problems that are of vital importance in many areas of the applied sciences. In both inverse spectral theory and inverse scattering theory, one tries to determine a coefficient in a differential equation from measurements of either the eigenvalues of the problem or the field "far away" from the scatterer. We hope that these two examples clearly show that a successful treatment of nonlinear inverse problems requires a solid knowledge of characteristic features of the corresponding direct problem. The combination of classical analysis and modern areas of applied and numerical analysis is, in the author's opinion, one of the fascinating features of this relatively new area of applied mathematics.

This book arose from a number of graduate courses, lectures, and survey talks during my time at the universities of Göttingen and Erlangen/Nürnberg. It was my intention to present a fairly elementary and complete introduction to the field of inverse problems, accessible not only to mathematicians but also to physicists and engineers. I tried to include as many proofs as possible as long as they required knowledge only of classical differential and integral calculus. The notions of functional analysis make it possible to treat different kinds of inverse problems in a common language and extract its basic features. For the convenience of the reader, I have collected the basic definitions and theorems from linear and nonlinear functional analysis at the end of the book in an appendix. Results on nonlinear mappings, in particular for the Fréchet derivative, are only needed in Chapters 4 and 5.

The book is organized as follows: In Chapter 1, we begin with a list of pairs of direct and inverse problems. Many of them are quite elementary and should be well-known. We formulate them from the point of view of inverse theory to demonstrate that the study of particular inverse problems has a long history. Sections 1.3 and 1.4 introduce the notions of ill-posedness and the worst-case error. While ill-posedness of a problem (roughly speaking) implies that the solution cannot be computed numerically – which is a very pessimistic point of view – the notion of the worst-case error leads to the possibility that stability can be recovered if additional information is available. We illustrate these notions with several elementary examples.

In Chapter 2, we study the general regularization theory for linear ill-posed equations in Hilbert spaces. The general concept in Section 2.1 is followed by the most important special examples: Tikhonov regularization in Section 2.2, Landweber iteration in Section 2.3, and spectral cutoff in Section 2.4. These regularization methods are applied to a test example in Section 2.5. While in Sections 2.1–2.5 the regularization parameter has been chosen a priori, i.e., before starting the actual computation, Sections 2.6–2.8 are devoted to regularization methods in which the regularization parameter is chosen implicitly by the stopping rule of the algorithm. In Sections 2.6 and 2.7, we study Morozov's discrepancy principle and, again, Landweber's iteration method. In contrast to these *linear* regularization schemes, we will investigate the conjugate gradient method in Section 2.8. This algorithm can be interpreted as a *nonlinear* regularization method and is much more difficult to analyze.

Chapter 2 deals with ill-posed problems in infinite-dimensional spaces. However, in practical situations, these problems are first discretized. The discretization of linear ill-posed problems leads to badly conditioned finite linear systems. This subject will be treated in Chapter 3. In Section 3.1, we recall basic facts about general projection methods. In Section 3.2, we will study several Galerkin methods as special cases and apply the results to Symm's integral equation in Section 3.3. This equation serves as a popular model equation in many papers on the numerical treatment of integral equations of the first kind with weakly singular kernels. We will present a complete and elementary existence and uniqueness theory of this equation in Sobolev spaces and apply the results about Galerkin methods to this equation. In Section 3.4, we study collocation methods. Here, we restrict ourselves to two examples: the moment collocation and the collocation of Symm's integral equation with trigonometric polynomials or piecewise constant functions as basis functions. In Section 3.5, we compare the different regularization techniques for a concrete numerical example of Symm's integral equation. Chapter 3 is completed by an investigation of the Backus–Gilbert method. Although this method does not quite fit into the general regularization theory, it is nevertheless widely used in the applied sciences to solve moment problems.

In Chapter 4, we study an *inverse eigenvalue problem* for a linear ordinary differential equation of second order. In Sections 4.2 and 4.3, we develop a careful analysis of the direct problem, which includes the asymptotic behavior of the eigenvalues and eigenfunctions. Section 4.4 is devoted to the question of uniqueness of the inverse problem, i.e., the problem of recovering the coefficient in the differential equation from the knowledge of one or two spectra. In Section 4.5, we show that this inverse problem is closely related to a parameter identification problem for parabolic equations. Section 4.6 describes some numerical reconstruction techniques for the inverse spectral problem.

In Chapter 5, we introduce the reader to the field of *inverse scattering theory*. Inverse scattering problems occur in several areas of science and technology, such as medical imaging, nondestructive testing of material, and geological prospecting. In Section 5.2, we study the direct problem and prove uniqueness, existence, and continuous dependence on the data. In Section 5.3, we study the asymptotic form of the scattered field as $r \to \infty$ and introduce the *far field pattern*. The corresponding inverse scattering problem is to recover the *index of refraction* from a knowledge of the far field pattern. We give a complete proof of uniqueness of this inverse problem in Section 5.4. Finally, Section 5.5 is devoted to the study of some recent reconstruction techniques for the inverse scattering problem.

Chapter 5 differs from previous ones in the unavoidable fact that we have to use some results from scattering theory without giving proofs. We will only formulate these results, and for the proofs we refer to easily accessible standard literature.

There exists a tremendous amount of literature on several aspects of inverse theory ranging from abstract regularization concepts to very concrete applications. Instead of trying to give a complete list of all relevant contributions, I mention only the monographs [15, 81, 86, 109, 130, 136, 137, 138, 144, 157, 158, 174, 215, 216], the proceedings, [5, 29, 53, 70, 93, 172, 192, 212], and survey articles [67, 116, 119, 122, 173] and refer to the references therein.

This book would not have been possible without the direct or indirect contributions of numerous colleagues and students. But, first of all, I would like to thank my father for his ability to stimulate my interest and love of mathematics during all the years. Also, I am deeply indebted to my friends and teachers, Professor Dr. Rainer Kress and Professor David Colton, who introduced me to the field of scattering theory and influenced my mathematical life in an essential way. This book is dedicated to my long friendship with them!

Particular thanks are given to Dr. Frank Hettlich, Dr. Stefan Ritter, and Dipl.-Math. Markus Wartha for carefully reading the manuscript. Furthermore, I would like to thank Professor William Rundell and Dr. Martin Hanke for their manuscripts on inverse Sturm–Liouville problems and conjugate gradient methods, respectively, on which parts of Chapters 4 and 2 are based.

Karlsruhe, April 1996 Andreas Kirsch

Contents

1

Introduction and Basic Concepts

1.1 Examples of Inverse Problems

In this section, we present some examples of pairs of problems that are inverse to each other. We start with some simple examples that are normally not even recognized as inverse problems. Most of them are taken from the survey article [119] and the monograph [87].

Example 1.1
Find a polynomial p of degree n with given zeros $x_1, \dots, x_n$. This problem is inverse to the direct problem: Find the zeros $x_1, \dots. x_n$ of a given polynomial p. In this example, the inverse problem is easier to solve. Its solution is $p(x) = c(x - x_1) \dots (x - x_n)$ with an arbitrary constant c.

Example 1.2
Find a polynomial p of degree n that assumes given values $y_1, \dots, y_n \in \mathbb{R}$ at given points $x_1, \dots, x_n \in \mathbb{R}$. This problem is inverse to the direct problem of calculating the given polynomial at given $x_1, \dots, x_n$. The inverse problem is the *Lagrange interpolation problem.*

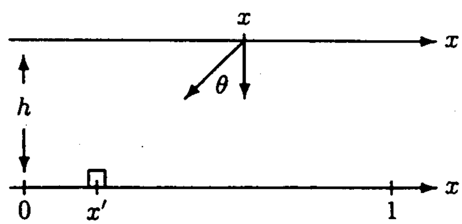
Example 1.3
Given a real symmetric $n \times n$ matrix A and n real numbers $\lambda_1, \dots, \lambda_n$, find a diagonal matrix D such that $A + D$ has the eigenvalues $\lambda_1, \dots, \lambda_n$. This problem is inverse to the direct problem of computing the eigenvalues of the given matrix $A + D$.

Example 1.4
This inverse problem is used on intelligence tests: Given the first few terms $a_1, a_2, \dots, a_k$ of a sequence, find the law of formation of the sequence, i.e., find a_n for all n! Usually, only the next two or three terms are asked for to show that the law of formation has been found. The corresponding direct problem is to evaluate the sequence (a_n) given the law of formation. It is clear that such inverse problems always have many solutions (from the mathematical point of view), and for this reason their use on intelligence tests has been criticized.

Example 1.5 *(Geological prospecting)*
In general, this is the problem of determining the location, shape, and/or some parameters (such as conductivity) of geological anomalies in Earth's interior from measurements at its surface. We consider a simple one-dimensional example and describe the following inverse problem.

Determine changes $\rho = \rho(x)$, $0 \le x \le 1$, of the mass density of an anomalous region at depth h from measurements of the vertical component $f_v(x)$ of the change of force at x. $\rho(x')\Delta x'$ is the mass of a "volume element" at x' and $\sqrt{(x-x')^2+h^2}$ is its distance from the instrument. The change of gravity is described by Newton's law of gravity $f = \gamma \frac{m}{r^2}$ with gravitational constant γ. For the vertical component, we have

$$\Delta f_v(x) = \gamma \frac{\rho(x')\Delta x'}{(x-x')^2+h^2} \cos\theta = \gamma \frac{h\,\rho(x')\Delta x'}{\left[(x-x')^2+h^2\right]^{3/2}}.$$



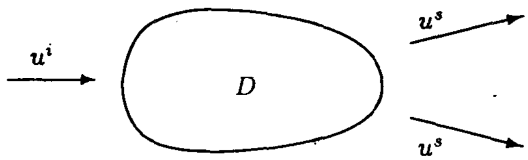
This yields the following integral equation for the determination of ρ:

$$f_v(x) = \gamma\, h \int_0^1 \frac{\rho(x')}{\left[(x-x')^2+h^2\right]^{3/2}}\, dx' \quad \text{for } 0 \le x \le 1. \qquad (1.1)$$

We refer to [4, 81, 229] for further reading on this and related inverse problems in geological prospecting.

Example 1.6 *(Inverse scattering problem)*
Find the shape of a scattering object, given the intensity (and phase) of

sound or electromagnetic waves scattered by this object. The corresponding direct problem is that of calculating the scattered wave for a given object.



More precisely, the *direct problem* can be described as follows. Let a bounded region $D \subset \mathbb{R}^N$ ($N = 2$ or 3) be given with smooth boundary ∂D (the scattering object) and a plane *incident* wave $u^i(x) = e^{ik\hat{\theta}\cdot x}$, where $k > 0$ denotes the wave number and $\hat{\theta}$ is a unit vector that describes the direction of the incident wave. The direct problem is to find the *total field* $u = u^i + u^s$ as the sum of the incident field u^i and the *scattered field* u^s such that

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^N \setminus \overline{D}, \qquad u = 0 \quad \text{on } \partial D, \tag{1.2a}$$

$$\frac{\partial u^s}{\partial r} - iku^s = \mathcal{O}\big(r^{-(N+1)/2}\big) \quad \text{for } r = |x| \to \infty \text{ uniformly in } \frac{x}{|x|}. \tag{1.2b}$$

For *acoustic* scattering problems, $v(x,t) = u(x)e^{-i\omega t}$ describes the pressure and $k = \omega/c$ is the wave number with speed of sound c. For suitably polarized time harmonic *electromagnetic* scattering problems, Maxwell's equations reduce to the *two-dimensional Helmholtz equation* $\Delta u + k^2 u = 0$ for the components of the electric (or magnetic) field u. The wave number k is given in terms of the dielectric constant ε and permeability μ by $k = \sqrt{\varepsilon\mu}\,\omega$.

In both cases, the radiation condition (1.2b) yields the following asymptotic behavior:

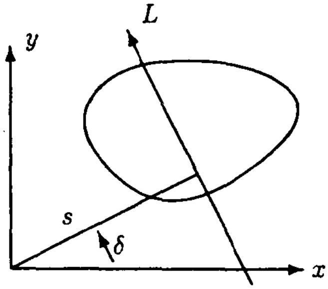
$$u^s(x) = \frac{\exp(ik|x|)}{|x|^{(N-1)/2}} u_\infty(\hat{x}) + \mathcal{O}\big(|x|^{-(N+1)/2}\big) \quad \text{as } |x| \to \infty,$$

where $\hat{x} = x/|x|$. The *inverse problem* is to determine the shape of D when the *far field pattern* $u_\infty(\hat{x})$ is measured for all $\hat{x}$ on the unit sphere in $\mathbb{R}^N$.

These and related inverse scattering problems have various applications in computer tomography, seismic and electromagnetic exploration in geophysics, and nondestructive testing of materials, for example. An inverse scattering problem of this type will be treated in detail in Chapter 5.

Standard literature on these direct and inverse scattering problems are the monographs [37, 38, 139] and the survey articles [34, 203].

Example 1.7 *(Computer tomography)*
The most spectacular application of the Radon transform is in medical imaging. For example, consider a fixed plane through a human body. Let $\rho(x, y)$ denote the change of density at the point (x, y), and let L be any line in the plane. Suppose that we direct a thin beam of X-rays into the body along L and measure how much the intensity is attenuated by going through the body.



Let L be parametrized by (s, δ), where $s \in \mathbb{R}$ and $\delta \in [0, \pi)$. The ray $L_{s,\delta}$ has the coordinates

$$se^{i\delta} + iue^{i\delta} \in \mathbb{C}, \quad u \in \mathbb{R},$$

where we have identified $\mathbb{C}$ with $\mathbb{R}^2$. The attenuation of the intensity I is approximately described by $dI = -\gamma\rho I\, du$ with some constant γ. Integration along the ray yields

$$\ln I(u) = -\gamma \int_{u_0}^{u} \rho\left(se^{i\delta} + iue^{i\delta}\right) du$$

or, assuming that ρ is of compact support, the relative intensity loss is given by

$$\ln I(\infty) = -\gamma \int_{-\infty}^{\infty} \rho\left(se^{i\delta} + iue^{i\delta}\right) du.$$

In principle, from the attenuation factors we can compute all line integrals

$$(R\rho)(s, \delta) := \int_{-\infty}^{\infty} \rho\left(se^{i\delta} + iue^{i\delta}\right) du, \quad s \in \mathbb{R},\ \delta \in [0, \pi). \tag{1.3}$$

$R\rho$ is called the *Radon transform* of ρ. The *direct problem* is to compute the Radon transform $R\rho$ when ρ is given. The *inverse problem* is to determine the density ρ for a given Radon transform $R\rho$ (i.e., measurements of all line integrals).

The problem simplifies in the following special case, where we assume that ρ is radially symmetric and we choose only vertical rays. Then $\rho = \rho(r)$, $r = \sqrt{x^2 + y^2}$, and the ray L_x passing through $(x, 0)$ can be parametrized by (x, u), $u \in \mathbb{R}$. This leads to (the factor 2 is due to symmetry)

$$V(x) \;:=\; \ln I(\infty) \;=\; -2\gamma \int_0^\infty \rho\left(\sqrt{x^2 + u^2}\right) du.$$

Again, we assume that ρ is of compact support in $\{x : |x| \le R\}$. The change of variables $u = \sqrt{r^2 - x^2}$ leads to

$$V(x) \;=\; -2\gamma \int_x^\infty \frac{r}{\sqrt{r^2 - x^2}}\, \rho(r)\, dr \;=\; -2\gamma \int_x^R \frac{r}{\sqrt{r^2 - x^2}}\, \rho(r)\, dr. \qquad (1.4)$$

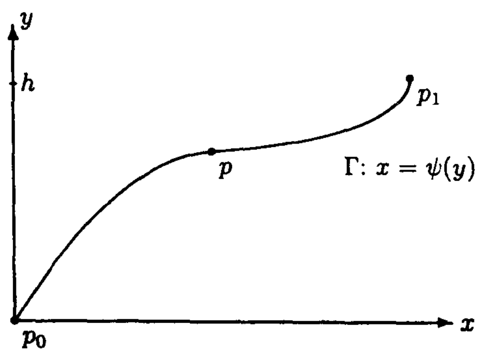
A further change of variables $z = R^2 - r^2$ and $y = R^2 - x^2$ transforms this equation into the following *Abel's integral equation* for the function $z \mapsto \rho(\sqrt{R^2 - z})$:

$$V\left(\sqrt{R^2 - y}\right) \;=\; -\gamma \int_0^y \frac{\rho\left(\sqrt{R^2 - z}\right)}{\sqrt{y - z}}\, dz, \quad 0 \le y \le R. \qquad (1.5)$$

The standard mathematical literature on the Radon transform and its applications are the monographs [102, 104, 166]. We refer also to the survey articles [105, 145, 147, 152].

The following example is due to Abel himself.

Example 1.8 *(Abel's integral equation)*
Let a mass element move along a curve Γ from a point p_1 on level $h > 0$ to a point p_0 on level $h = 0$. The only force acting on this mass element is the gravitational force mg.



The *direct problem* is to determine the time T in which the element moves from p_1 to p_0 when the curve Γ is given. In the *inverse problem*, one measures the time $T = T(h)$ for several values of h and tries to determine the curve Γ. Let the curve be parametrized by $x = \psi(y)$. Let p have the coordinates $(\psi(y), y)$.

By conservation of energy, i.e.,

$$E + U = \frac{m}{2}v^2 + m\,g\,y = \text{const} = m\,g\,h,$$

we conclude for the velocity that

$$\frac{ds}{dt} = v = \sqrt{2g(h-y)}.$$

The total time T from p_1 to p_0 is

$$T = T(h) = \int_{p_0}^{p_1} \frac{ds}{v} = \int_0^h \sqrt{\frac{1+\psi'(y)^2}{2g\,(h-y)}}\,dy \quad \text{for } h > 0.$$

Set $\varphi(y) = \sqrt{1+\psi'(y)^2}$ and let $f(h) := T(h)\sqrt{2g}$ be known (measured). Then we have to determine the unknown function φ from Abel's integral equation

$$\int_0^h \frac{\varphi(y)}{\sqrt{h-y}}\,dy = f(h) \quad \text{for } h > 0. \tag{1.6}$$

A similar – but more important – problem occurs in seismology. One studies the problem to determine the velocity distribution c of Earth from measurements of the travel times of seismic waves (see [22]).

For further examples of inverse problems leading to Abel's integral equations, we refer to the lecture notes by R. Gorenflo and S. Vessella [84], the monograph [158], and the papers [141, 222].

Example 1.9 *(Backwards heat equation)*
Consider the one-dimensional heat equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} \tag{1.7a}$$

with boundary conditions

$$u(0,t) = u(\pi,t) = 0, \quad t \geq 0, \tag{1.7b}$$

and initial condition

$$u(x,0) = u_0(x), \quad 0 \leq x \leq \pi. \tag{1.7c}$$

Separation of variables leads to the solution

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-n^2 t} \sin(nx) \quad \text{with} \quad a_n = \frac{2}{\pi} \int_0^{\pi} u_0(y) \sin(ny) dy. \tag{1.8}$$

The *direct problem* is to solve the classical initial boundary value problem: Given the initial temperature distribution u_0 and the final time T, determine $u(\cdot, T)$. In the *inverse problem*, one measures the final temperature distribution $u(\cdot, T)$ and tries to determine the temperature at earlier times $t < T$, e.g., the initial temperature $u(\cdot, 0)$.

From solution formula (1.8), we see that we have to determine $u_0 := u(\cdot, 0)$ from the following integral equation:

$$u(x,T) = \frac{2}{\pi} \int_0^{\pi} k(x,y) u_0(y)\, dy, \quad 0 \le x \le \pi, \tag{1.9}$$

where

$$k(x,y) := \sum_{n=1}^{\infty} e^{-n^2 T} \sin(nx) \sin(ny). \tag{1.10}$$

We refer to the monographs [15, 138, 158] and papers [24, 31, 33, 59, 60, 72, 153, 202] for further reading on this subject.

Example 1.10 *(Diffusion in inhomogeneous medium)*
The equation of diffusion in an inhomogeneous medium (now in two dimensions) is described by the equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{1}{c} \operatorname{div}\big(\kappa \operatorname{grad} u(x,t)\big), \quad x \in D,\ t > 0, \tag{1.11}$$

where c is a constant and $\kappa = \kappa(x)$ is a parameter describing the medium. In the stationary case, this reduces to

$$\operatorname{div}(\kappa \operatorname{grad} u) = 0 \quad \text{in } D. \tag{1.12}$$

The *direct problem* is to solve the boundary value problem for this equation for given boundary values $u|_{\partial D}$ and given function κ. In the *inverse problem*, one measures u and the flux $\frac{\partial u}{\partial \nu}$ on the boundary ∂D and tries to determine the unknown function κ in D.

This is an example of a *parameter identification problem* for a partial differential equation. Among the extensive literature on parameter identification problems, we only mention the classical papers [128, 183, 182], the monographs [13, 15, 158], and the survey article [160].

Example 1.11 *(Sturm-Liouville eigenvalue problem)*
Let a string of length L and mass density $\rho = \rho(x) > 0$, $0 \le x \le L$, be fixed at the endpoints $x = 0$ and $x = L$. Plucking the string produces tones due to vibrations. Let $v(x,t)$, $0 \le x \le L$, $t > 0$, be the displacement at x and time t. It satisfies the *wave equation*

$$\rho(x)\,\frac{\partial^2 v(x,t)}{\partial t^2} = \frac{\partial^2 v(x,t)}{\partial x^2}, \quad 0 < x < L, \; t > 0, \tag{1.13}$$

subject to boundary conditions $v(0,t) = v(L,t) = 0$ for $t > 0$.

A periodic displacement of the form

$$v(x,t) = w(x)\left[a\cos\omega t + b\sin\omega t\right]$$

with frequency $\omega > 0$ is called a *pure tone*. This form of v solves the boundary value problem (1.13) if and only if w and ω satisfy the Sturm–Liouville eigenvalue problem

$$w''(x) + \omega^2\rho(x)\,w(x) = 0, \; 0 < x < L, \quad w(0) = w(L) = 0. \tag{1.14}$$

The *direct problem* is to compute the eigenfrequencies ω and the corresponding eigenfunctions for known function ρ. In the *inverse problem*, one tries to determine the mass density ρ from a number of measured frequencies ω.

We will see in Chapter 4 that parameter estimation problems for parabolic and hyperbolic initial boundary value problems are closely related to inverse spectral problems.

Example 1.12 *(Inverse Stefan problem)*
The physicist Stefan (see [207]) modeled the melting of arctic ice in the summer by a simple one-dimensional model. In particular, consider a homogeneous block of ice filling the region $x \ge \ell$ at time $t = 0$. The ice starts to melt by heating the block at the left end. Thus, at time $t > 0$ the region between $x = 0$ and $x = s(t)$ for some $s(t) > 0$ is filled with water and the region $x \ge s(t)$ is filled with ice.