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Friedrich Hirzebruch

Topological Methods in Algebraic Geometry

代数几何中的拓扑方法

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Preface to the first edition

In recent years new topological methods, especially the theory of sheaves founded by J. LERAY, have been applied successfully to algebraic geometry and to the theory of functions of several complex variables. H. CARTAN and J.-P. SERRE have shown how fundamental theorems on holomorphically complete manifolds (STEIN manifolds) can be formulated in terms of sheaf theory. These theorems imply many facts of function theory because the domains of holomorphy are holomorphically complete. They can also be applied to algebraic geometry because the complement of a hyperplane section of an algebraic manifold is holomorphically complete. J.-P. SERRE has obtained important results on algebraic manifolds by these and other methods. Recently many of his results have been proved for algebraic varieties defined over a field of arbitrary characteristic. K. KODAIRA and D. C. SPENCER have also applied sheaf theory to algebraic geometry with great success. Their methods differ from those of SERRE in that they use techniques from differential geometry (harmonic integrals etc.) but do not make any use of the theory of STEIN manifolds. M. F. ATIYAH and W. V. D. HODGE have dealt successfully with problems on integrals of the second kind on algebraic manifolds with the help of sheaf theory.

I was able to work together with K. KODAIRA and D. C. SPENCER during a stay at the Institute for Advanced Study at Princeton from 1952 to 1954. My aim was to apply, alongside the theory of sheaves, the theory of characteristic classes and the new results of R. THOM on differentiable manifolds. In connection with the applications to algebraic geometry I studied the earlier research of J. A. TODD. During this time at the Institute I collaborated with A. BOREL, conducted a long correspondence with THOM and was able to see the correspondence of KODAIRA and SPENCER with SERRE. I thus received much stimulating help at Princeton and I wish to express my sincere thanks to A. BOREL, K. KODAIRA, J.-P. SERRE, D. C. SPENCER and R. THOM.

This book grew out of a manuscript which was intended for publication in a journal and which contained an exposition of the results obtained during my stay in Princeton. Professor F. K. SCHMIDT invited me to use it by writing a report for the "Ergebnisse der Mathematik". Large parts of the original manuscript have been taken over unchanged, while other parts of a more expository nature have been expanded. In this way the book has become a mixture between a report, a textbook and an original article. I wish to thank Professor F. K. SCHMIDT for his great interest in my work.

I must thank especially the Institute for Advanced Study at Princeton for the award of a scholarship which allowed me two years of undisturbed work in a particularly stimulating mathematical atmosphere. I wish to thank the University of Erlangen which gave me leave of absence during this period and which has supported me in every way; the Science Faculty of the University of Münster, especially Professor H. BEHNKE, for accepting this book as a Habilitationsschrift; and the Society for the Advancement of the University of Münster for financial help during the final preparation of the manuscript. I am indebted to R. REMMERT and G. SCHEJA for their help with the proofs, and to H.-J. NASTOLD for preparing the index. Last, but not least, I wish to thank the publishers who have generously complied with all my wishes.

Fine Hall, Princeton F. HIRZEBRUCH

23 January 1956

Preface to the third edition

In the ten years since the publication of the first edition, the main results have been extended in several directions. On the one hand the RIEMANN-ROCH theorem for algebraic manifolds has been generalised by GROTHENDIECK to a theorem on maps of projective algebraic varieties over a ground field of arbitrary characteristic. On the other hand ATIYAH and SINGER have proved an index theorem for elliptic differential operators on differentiable manifolds which includes, as a special case, the RIEMANN-ROCH theorem for arbitrary compact complex manifolds.

There has been a parallel development of the integrality theorems for characteristic classes. At first these were proved for differentiable manifolds by complicated deductions from the almost complex and algebraic cases. Now they can be deduced directly from theorems on maps of compact differentiable manifolds which are analogous to the RIEMANN-ROCH theorem of GROTHENDIECK. A basic tool is the ring $K(X)$ formed from the semi-ring of all isomorphism classes of complex vector bundles over a topological space X, together with the BOTT periodicity theorem which describes $K(X)$ when X is a sphere. The integrality theorems also follow from the ATIYAH-SINGER index theorem in the same way that the integrality of the TODD genus for algebraic manifolds follows from the RIEMANN-ROCH theorem.

Very recently ATIYAH and BOTT obtained fixed point theorems of the type first proved by LEFSCHETZ. A holomorphic map of a compact complex manifold V operates, under certain conditions, on the cohomology groups of V with coefficients in the sheaf of local holomorphic sections of a complex analytic vector bundle W over V. For a special class of holomorphic maps, ATIYAH and BOTT express the alternating sum of the traces of these operations in terms of the fixed point set of the map. For the identity map this reduces to the RIEMANN-ROCH theorem. Another application yields the formulae of LANGLANDS (see 22.3) for the dimensions of spaces of automorphic forms. ATIYAH and BOTT carry out these investigations for arbitrary elliptic operators and differentiable maps, obtaining a trace formula which generalises the index theorem. Their results have a topological counterpart which generalises the integrality theorems.

The aim of the translation has been to take account of these developments – especially those which directly involve the TODD genus – within the framework of the original text. The translator has done this chiefly by the addition of bibliographical notes to each chapter and by a new appendix containing a survey, mostly without proofs, of some of the applications and generalisations of the RIEMANN-ROCH theorem made since 1956. The fixed point theorems of ATIYAH and BOTT could be mentioned only very briefly, since they became known after the manuscript for the appendix had been finished. A second appendix consists of a paper by A. BOREL which was quoted in the first edition but which has not previously been published. Certain amendments to the text have been made in order to increase the usefulness of the book as a work of reference. Except for Theorems 2.8.4, 2.9.2, 2.11.2, 4.11.1–4.11.4, 10.1.1, 16.2.1 and 16.2.2 in the new text, all theorems are numbered as in the first edition.

The author thanks R. L. E. SCHWARZENBERGER for his efficient work in translating and editing this new edition, and for writing the new appendix, and A. BOREL for allowing his paper to be added to the book.

We are also grateful to Professor F. K. SCHMIDT for suggesting that this edition should appear in the "Grundlehren der mathematischen Wissenschaften", to D. ARLT, E. BRIESKORN and K. H. MAYER for checking the manuscript, and to ANN GARFIELD for preparing the typescript. Finally we wish to thank the publishers for their continued cooperation.

Bonn and Coventry
23 January 1966

F. HIRZEBRUCH
R. L. E. SCHWARZENBERGER

Contents

Introduction

The theory of sheaves, developed and applied to various topological problems by LERAY [1], [2][1]), has recently been applied to algebraic geometry and to the theory of functions of several complex variables. These applications, due chiefly to CARTAN, SERRE, KODAIRA, SPENCER, ATIYAH and HODGE have made possible a common systematic approach to both subjects. This book makes a further contribution to this development for algebraic geometry. In addition it contains applications of the results of THOM on cobordism of differentiable manifolds which are of independent interest. Sheaf theory and cobordism theory together provide the foundations for the present results on algebraic manifolds. This introduction gives an outline (0.1–0.8) of the results in the book. It does not contain precise definitions; these can be found by reference to the index. Remarks on terminology and notations used throughout the book are at the end of the introduction (0.9).

0.1. A compact complex manifold V (not necessarily connected) is called an *algebraic manifold* if it admits a complex analytic embedding as a submanifold of a complex projective space of some dimension. By a theorem of CHOW [1] this definition is equivalent to the classical definition of a non-singular algebraic variety. Algebraic manifolds in this sense are often also called non-singular projective varieties. In 0.1–0.6 we consider only algebraic manifolds.

Let V_n be an algebraic manifold of complex dimension n. The arithmetic genus of V_n has been defined in four distinct ways. The postulation formula (HILBERT characteristic function) can be used to define integers $p_a(V_n)$ and $P_a(V_n)$. These are the first two definitions. SEVERI conjectured that

$$p_a(V_n) = P_a(V_n) = g_n - g_{n-1} + \cdots + (-1)^{n-1} g_1 , \qquad (1)$$

where g_i is the number of complex-linearly independent holomorphic differential forms on V_n of degree i (i-fold differentials of the first kind). The alternating sum of the g_i can be regarded as a third definition of the arithmetic genus. Further details can be found, for instance, in SEVERI [1]. Equation (1) can be proved easily by means of sheaf theory (KODAIRA-SPENCER [1]) and therefore the first three definitions of the arithmetic genus agree.

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[1]) Numbers in square brackets refer to the bibliography at the end of the book.

The form of the alternating sum of g_i in (1) is inconvenient and we modify the classical definition slightly. We call

$$\chi(V_n) = \sum_{i=0}^{n} (-1)^i g_i \tag{2}$$

the *arithmetic genus* of the algebraic manifold V_n. The integer g_0 in (2) is the number of linearly independent holomorphic functions on V_n and is therefore equal to the number of connected components of V_n. It is usual to call g_n the *geometric genus* of V_n and g_1 the *irregularity* of V_n. In the case $n = 1$ a connected algebraic curve V_1 is a compact RIEMANN surface homeomorphic to a sphere with p handles. Then $g_n = g_1 = p$ and the arithmetic genus of V_1 is $1 - p$. The arithmetic genus and the geometric genus behave multiplicatively:

The genus of the cartesian product $V \times W$ of two algebraic manifolds is the product of the genus of V and the genus of W.

Under the old terminology the arithmetic genus clearly does not have this property. The arithmetic genus $\chi(V_n)$ is a birational invariant because each g_i is a birational invariant (KÄHLER [1] and VAN DER WAERDEN [1], [2]). Under the old terminology the arithmetic genus of a rational variety is 0. According to the present definition it is 1.

0.2. The fourth definition of the arithmetic genus is due to TODD [1]. He showed in 1937 that the arithmetic genus could be represented in terms of the canonical classes of EGER-TODD (TODD [3]). The proof is however incomplete: it relies on a lemma of SEVERI for which no complete proof exists in the literature.

The EGER-TODD class K_i of V_n is by definition an equivalence class of algebraic cycles of real dimension $2n - 2i$. The equivalence relation implies, but does not in general coincide with, the relation of homology equivalence. For example K_1 $(= K)$ is the class of canonical divisors of V_n. (A divisor is canonical if it is the divisor of a meromorphic n-form.) The equivalence relation for $i = 1$ is linear equivalence of divisors. The class K_i defines a $(2n - 2i)$-dimensional homology class. This determines a $2i$-dimensional cohomology class which agrees (up to sign) with the CHERN class c_i of V_n. This "agreement" between the EGER-TODD classes and the CHERN classes was proved by NAKANO [2] (see also CHERN [2], HODGE [3] and ATIYAH [3]).

Remark: The sign of the $2i$-dimensional cohomology class determined by K_i depends on the orientation of V_n. We shall always use the natural orientation of V_n. If $z_1, z_2, \ldots, z_n$ are local coordinates with $z_k = x_k + i\,y_k$ then this orientation is given by the ordering $x_1, y_1, x_2, y_2, \ldots, x_n, y_n$ or in other words by the positive volume element $dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge \cdots \wedge dx_n \wedge dy_n$. In this case K_i determines the cohomology class $(-1)^i c_i$.

In this book we use only the CHERN *classes* and so the fact that the EGER-TODD classes agree with the CHERN classes is not needed. The definition of the TODD genus $T(V_n)$ is given in terms of the CHERN classes. One of the chief purposes of this book is then to prove that $\chi(V_n) = T(V_n)$.

0.3. The natural orientation of V_n defines an element of the $2n$-dimensional integral homology group $H_{2n}(V_n, \mathbf{Z})$ called the fundamental cycle of V_n. The value of a $2n$-dimensional cohomology class b on the fundamental cycle is denoted by $b\,[V_n]$.

The definition of $T(V_n)$ is in terms of a certain polynomial T_n of weight n in the CHERN classes c_i of V_n, the products being taken in the cohomology ring of V_n. This polynomial is defined algebraically in § 1; it is a rational $2n$-dimensional cohomology class whose value on the fundamental cycle is by definition $T(V_n)$. For small n (see 1.7)

$$T(V_1) = \tfrac{1}{2}\, c_1\,[V_1], \quad T(V_2) = \tfrac{1}{12}\,(c_1^2 + c_2)\,[V_2], \quad T(V_3) = \tfrac{1}{24}\, c_1\, c_2\,[V_3]\,. \quad (3)$$

The definition implies that $T(V_n)$ is a rational number. The equation $\chi(V_n) = T(V_n)$ implies the non-trivial fact that $T(V_n)$ is an integer and that $T(V_n)$ is a birational invariant. The sequence of polynomials $\{T_n\}$ must be chosen so that, like the arithmetic genus, $T(V_n)$ behaves multiplicatively on cartesian products. There are many sequences with this property: it is sufficient for $\{T_n\}$ to be a multiplicative sequence (§ 1). The sequence $\{T_n\}$ must be further chosen so that $T(V_n)$ agrees with $\chi(V_n)$ whenever possible. In particular if $\mathbf{P}_n(\mathbf{C})$ denotes the n-dimensional complex projective space then $T(\mathbf{P}_n(\mathbf{C})) = 1$ for all n. This condition is used in § 1 to determine the multiplicative sequence $\{T_n\}$ uniquely (Lemma 1.7.1).

For fixed n the polynomial T_n is determined uniquely by the following property: $T_n\,[V_n] = 1$ *if* $V = \mathbf{P}_{j_1}(\mathbf{C}) \times \cdots \times \mathbf{P}_{j_r}(\mathbf{C})$ *is a cartesian product of complex projective spaces with* $j_1 + \cdots + j_r = n$. *Therefore* T_n *is the unique polynomial which takes the value* 1 *on all rational manifolds of dimension* n.

0.4. The divisors of the algebraic manifold V_n can be formed into equivalence classes with respect to linear equivalence. A divisor is linearly equivalent to zero if it is the divisor (f) of a meromorphic function f on V_n. This equivalence is compatible with addition of divisors and therefore the divisor classes form an additive group. We can also consider complex analytic line bundles (with fibre $\mathbf{C}$ and group $\mathbf{C}^*$; see 0.9) over V_n. In this introduction we identify isomorphic line bundles (see 0.9). Then the line bundles form an abelian group with respect to the tensor product $\otimes$. The identity element, denoted by $\mathbf{1}$, is the trivial complex line bundle $X \times \mathbf{C}$. The inverse of a complex line bundle F is denoted by F^{-1}. *The group of line bundles is isomorphic to the group of divisor classes:*

Every divisor determines a line bundle. The sum of two divisors determines the tensor product of the corresponding line bundles. Two divisors determine the same line bundle if and only if they are linearly equivalent. Finally, every line bundle is determined by some divisor (KODAIRA-SPENCER [2]). Denote by $H^0(V_n, D)$ the complex vector space of all meromorphic functions f on V_n such that $D + (f)$ is a divisor with no poles. $H^0(V_n, D)$ is the "RIEMANN-ROCH space" of D and is *finite dimensional*. The dimension $\dim H^0(V_n, D)$ depends only on the divisor class of D. The determination of $\dim H^0(V_n, D)$ for a given divisor D is the RIEMANN-ROCH problem. If F is the line bundle corresponding to the divisor D then $H^0(V_n, D)$ is isomorphic to $H^0(V_n, F)$, the complex vector space of holomorphic sections of F.

0.5. It has already been said that one aim of this work is to prove the equation

$$\chi(V_n) = T(V_n)\,. \tag{4}$$

The CHERN number $c_n[V_n]$ is equal to the EULER-POINCARÉ characteristic of V_n. Therefore equation (4) gives, for a connected algebraic curve V homeomorphic to a sphere with p handles:

$$\chi(V_1) = T(V_1) = \tfrac{1}{2}\, c_1[V_1] = \tfrac{1}{2}\,(2-2p)\,. \tag{4_1}$$

The RIEMANN-ROCH theorem for algebraic curves states (see for instance WEYL [1]):

$$\dim H^0(V_1, D) - \dim H^0(V_1, K - D) = d + 1 - p \tag{4_1^*}$$

where d is the degree of the divisor D and K is a canonical divisor of V_1. Since $\dim H^0(V_1, K) = g_1$ the substitution $D = 0$ in (4_1^*) gives (4_1). It will be shown that for algebraic manifolds of arbitrary dimension equation (4) admits a generalisation which corresponds precisely to the generalisation (4_1^*) of (4_1). This generalisation will be given in terms of line bundles rather than divisors.

Let F be a complex analytic line bundle and let $H^i(V_n, F)$ be the i-th cohomology group of V_n with coefficients in the sheaf of germs of local holomorphic sections of F. In the case $F = 1$ this is the sheaf of germs of local holomorphic functions. The cohomology "group" $H^i(V_n, F)$ is a complex vector space which, by results of CARTAN-SERRE [1] (see also CARTAN [4]) and KODAIRA [3], is of finite dimension. The vector space $H^0(V_n, F)$ is the "RIEMANN-ROCH space" of F defined in 0.4. A theorem of DOLBEAULT [1] implies that $\dim H^i(V_n, 1) = g_i$. The integer $\dim H^i(V_n, F)$ depends only on the isomorphism class of F and is zero for $i > n$. It is therefore possible to define

$$\chi(V_n, F) = \sum_{i=0}^{n} (-1)^i \dim H^i(V_n, F)\,. \tag{5}$$

This is the required generalisation of the left hand side of (4). It will be shown that $\chi(V_n, F)$ can be expressed as a certain polynomial in the CHERN classes of V_n and a 2-dimensional cohomology class f determined by the line bundle F. Here f is the first CHERN class of F (the cohomology obstruction to the existence of a continuous never zero section of F). If F is represented by a divisor D then f is also determined by the $(2n-2)$-dimensional homology class corresponding to D. For small n,

$$\chi(V_1, F) = \left(f + \tfrac{1}{2} c_1\right)[V_1], \quad \chi(V_2, F) = \left(\tfrac{1}{2}(f^2 + f c_1) + \tfrac{1}{12}(c_1^2 + c_2)\right)[V_2],$$

$$\chi(V_3, F) = \left(\tfrac{1}{6} f^3 + \tfrac{1}{4} f^2 c_1 + \tfrac{1}{12} f(c_1^2 + c_2) + \tfrac{1}{24} c_1 c_2\right)[V_3].$$

This is the generalisation of the RIEMANN-ROCH theorem to algebraic manifolds of arbitrary dimension (Theorem 20.3.2). By the SERRE duality theorem (see 15.4.2) $\dim H^1(V_1, F) = \dim H^0(V_1, K \otimes F^{-1})$ and $\dim H^2(V_2, F) = \dim H^0(V_2, K \otimes F^{-1})$ where K denotes the line bundle determined by canonical divisors. It follows that the equations for $\chi(V_1, F)$ and $\chi(V_2, F)$ imply the classical RIEMANN-ROCH theorem for an algebraic curve and for an algebraic surface. Full details are given in 19.2 and 20.7.

KODAIRA [4] and SERRE have given conditions under which $\dim H^i(V_n, F) = 0$ for $i > 0$ (see Theorem 18.2.2 and CARTAN [4], Exposé XVIII). The definition of $\chi(V_n, F)$ in (5) then shows that our formula for $\chi(V_n, F)$ yields a formula for $H^0(V_n, F)$. In such cases the "RIEMANN-ROCH problem" stated in 0.4 is completely solved. This corresponds for algebraic curves to the well known fact that the term $\dim H^0(V_1, K - D)$ in (4_1^*) is zero if $d > 2p - 2$.

0.6. There is a further generalisation of equation (4). Let W be a complex analytic vector bundle over V_n [with fibre $\mathbf{C}_q$ and group $\mathbf{GL}(q, \mathbf{C})$; see 0.9]. Let $H^i(V_n, W)$ be the i-th cohomology group of V_n with coefficients in the sheaf of germs of local holomorphic sections of W. Then $H^i(V_n, W)$ is again a complex vector space of finite dimension and $\dim H^i(V_n, W)$ is zero for $i > n$. It is therefore possible to define

$$\chi(V_n, W) = \sum_{i=0}^{n} (-1)^i \dim H^i(V_n, W). \tag{6}$$

It was conjectured by SERRE, in a letter to KODAIRA and SPENCER (29 September 1953), that $\chi(V_n, W)$ could be expressed as a polynomial in the CHERN classes of V_n and the CHERN classes of W. We shall obtain an explicit formula for the polynomial of $\chi(V_n, W)$. This is the RIEMANN-ROCH theorem for vector bundles (Theorem 21.1.1). A corollary in the case $n = 1$ (algebraic curves) is the generalisation of the RIEMANN-ROCH theorem due to WEIL [1]. Full details are given in 21.1.

The result on $\chi(V_n, W)$ can be applied to particular vector bundles over V_n. We define (see KODAIRA-SPENCER [3])

$$\chi^p(V_n) = \chi(V_n, \lambda^p T) \tag{7}$$

where $\lambda^p T$ is the vector bundle of covariant p-vectors of V_n. The CHERN classes of $\lambda^p T$ can be expressed in terms of the CHERN classes of V_n (Theorem 4.4.3). Therefore $\chi^p(V_n)$ is a polynomial of weight n in the CHERN classes of V_n. By a theorem of DOLBEAULT [1], $\dim H^q(V_n, \lambda^p T)$ is the number $h^{p,q}$ of complex-linearly independent harmonic forms on V_n of type (p, q). Therefore $\chi^p(V_n) = \sum_{q=0}^{n} (-1)^q h^{p,q}$. For example, in the case $n = 4$, there is an equation

$$\chi^1(V_4) = h^{1,0} - h^{1,1} + h^{1,2} - h^{1,3} + h^{1,4} = 4\chi(V_4) - \tfrac{1}{12}(2c_4 + c_3 c_1)[V_4]. \tag{8}$$

The sum $\sum_{p=0}^{n} \chi^p(V_n)$ is clearly zero for n odd. The alternating sum $\sum_{p=0}^{n} (-1)^p \chi^p(V_n)$ is by theorems of DE RHAM and HODGE equal to the EULER-POINCARÉ characteristic $c_n[V_n]$ of V_n. The polynomials for $\chi^p(V_n)$ have the same properties. HODGE [4] proved that for n even the sum $\sum_{p=0}^{n} \chi^p(V_n)$ is equal to the index of V_n. By definition the index of V_n is the signature (number of positive eigenvalues minus number of negative eigenvalues) of the bilinear symmetric form $xy[V_n]$ $(x, y \in H^n(V_n, \mathbf{R}))$ on the n-dimensional real cohomology group of V_n. Therefore the index of V_n is a polynomial in the CHERN classes of V_n. This polynomial can actually be expressed as a polynomial in the PONTRJAGIN classes of V_n and is therefore defined for any oriented differentiable manifold.

0.7. We have just remarked that the main result of this book [the expression of $\chi(V_n, W)$ as a certain polynomial in the CHERN classes of V_n and W] implies that the index of an algebraic manifold V_{2k} can be expressed as a polynomial in the PONTRJAGIN classes of V_{2k}. In fact this theorem is the starting point of our investigation. Let M^{4k} be an oriented differentiable manifold of real dimension $4k$. In this work "differentiable" always means "C^∞-differentiable" so that all partial derivatives exist and are continuous. The orientation of M^{4k} defines a $4k$-dimensional fundamental cycle. The value of a $4k$-dimensional cohomology class b on the fundamental cycle is denoted by $b[M^{4k}]$. In Chapter Two the cobordism theory of THOM is used to express the index $\tau(M^{4k})$ of M^{4k} as a polynomial of weight k in the PONTRJAGIN classes of M^{4k}. For example,

$$\tau(M^4) = \tfrac{1}{3} p_1 [M^4], \quad \tau(M^8) = \tfrac{1}{45}(7p_2 - p_1^2)[M^8]. \tag{9}$$

The formula for $\tau(M^4)$ was conjectured by Wu. The formulae for $\tau(M^4)$ and $\tau(M^8)$ were both proved by THOM [2]. A brief summary of the deduction of the formula for $\chi(V_n, W)$ from that for $\tau(M^{4k})$ can be found in HIRZEBRUCH [2].

0.8. The definitions in 0.1–0.6 were formulated only for algebraic manifolds. In the proof of the RIEMANN-ROCH theorem we make this restriction only when necessary. The index theorem described in 0.7 is proved in Chapter Two for arbitrary oriented differentiable manifolds. The main results of THOM on cobordism are quoted: the proofs, which make use of differentiable approximation theorems and algebraic homotopy theory, are outside the scope of this work.

In Chapter Three the formal theory of the TODD genus and of the associated polynomials is developed for arbitrary compact almost complex manifolds (T-theory). In particular we obtain an integrality theorem (14.3.2). This theorem has actually little to do with almost complex manifolds; its relation to subsequent integrality theorems for differentiable manifolds is discussed in the bibliographical note to Chapter Three and in the Appendix.

In Chapter Four the theory of the integers $\chi(V_n, W)$ is developed as far as possible for arbitrary compact complex manifolds (χ-theory). The necessary results on sheaf cohomology due to CARTAN, DOLBEAULT, KODAIRA, SERRE and SPENCER are described briefly. In the course of the proof it is necessary to assume first that V_n is a KÄHLER manifold. Finally, if V_n is an algebraic manifold, we are able to identify the χ-theory with the T-theory (RIEMANN-ROCH theorem for vector bundles; Theorem 21.1.1).

The Appendix contains a review of applications and generalisations of the RIEMANN-ROCH theorem. In particular it is now known that the identification of the χ-theory with the T-theory holds for any compact complex manifold V_n (see § 25).

The author has tried to make the book as independent of other works as is possible within a limited length. The necessary preparatory material on multiplicative sequences, sheaves, fibre bundles and characteristic classes can be found in Chapter One.

0.9. Remarks on notation and terminology

The following notations are used throughout the book.

Z: integers, **Q**: rational numbers, **R**: real numbers, **C**: complex numbers, $\mathbf{R}^q$: vector space over **R** of q-ples $(x_1, \ldots, x_q)$ of real numbers, $\mathbf{C}_q$: vector space over **C** of q-ples of complex numbers. $\mathbf{GL}(q, \mathbf{R})$ denotes the group of invertible $q \times q$ matrices (a_{ik}) with real coefficients a_{ik}, *i. e.* the group of automorphisms of $\mathbf{R}^q$

$$x_i' = \sum_{k=1}^{q} a_{ik} x_k .$$

$\mathbf{GL}^+(q, \mathbf{R})$ denotes the subgroup of $\mathbf{GL}(q, \mathbf{R})$ consisting of matrices with positive determinant (the group of orientation preserving automorphisms). $\mathbf{O}(q)$ denotes the subgroup of orthogonal matrices of $\mathbf{GL}(q, \mathbf{R})$ and $\mathbf{SO}(q) = \mathbf{O}(q) \cap \mathbf{GL}^+(q, \mathbf{R})$. Similarly $\mathbf{GL}(q, \mathbf{C})$ denotes the group of invertible $q \times q$ matrices with complex coefficients, and $\mathbf{U}(q)$ the subgroup of unitary matrices of $\mathbf{GL}(q, \mathbf{C})$. We write $\mathbf{C}^* = \mathbf{GL}(1, \mathbf{C})$, the multiplicative group of non-zero complex numbers. $\mathbf{P}_{q-1}(\mathbf{C})$ denotes the complex projective space of complex dimension $q-1$ (the space of complex lines through the origin of $\mathbf{C}_q$). We shall often denote real dimension by an upper suffix (for example M^{4k}, $\mathbf{R}^q$) and complex dimension by a lower suffix (for example V_n, $\mathbf{C}_q$).

We have adopted one slight modification of the usual terminology. An isomorphism class of principal fibre bundles with structure group G is called a G-bundle. Thus a G-bundle is an element of a certain cohomology set. On the other hand, we use the words fibre bundle, line bundle and vector bundle to mean a particular fibre space and not an isomorphism class of such spaces (see 3.2). In Chapter Four all constructions depend only on the isomorphism class of the vector bundles involved and it is possible to drop this distinction (see 15.1).

The book is divided into chapters and then into paragraphs, which are numbered consecutively throughout the book. Formulae are numbered consecutively within each paragraph. The paragraphs are divided into sections. Thus 4.1 means section 1 of § 4; 4.1 (5) means formula (5) of § 4, which occurs in section 4.1; 4.1.1 refers to Theorem 1 of section 4.1. The index includes references to the first occurrence of any symbol.

Chapter One

Preparatory material

The elementary and formal algebraic theory of multiplicative sequences is contained in § 1. In particular the TODD polynomials T_j, and also the polynomials L_j used in the index theorem, are defined. Results on sheaves needed in the sequel are collected in § 2. The basic properties of fibre bundles are given in § 3. In § 4 these are applied to obtain characteristic classes. In particular, the CHERN classes and PONTRJAGIN classes are defined. The results of § 1 are not used until § 8. The reader is therefore advised to begin with § 2 and to refer to § 1 only when necessary.

§ 1. Multiplicative sequences

1.1. Let B be a commutative ring with identity element 1. Let $p_0 = 1$ and let $p_1, p_2, \ldots$ be indeterminates. Consider the ring $\mathfrak{B} = B[p_1, p_2, \ldots]$ obtained by adjoining the indeterminates p_i to B. Then $\mathfrak{B}$ is the ring of polynomials in the p_i with coefficients in B, and is graded in the following way:

The product $p_{j_1} p_{j_2} \ldots p_{j_r}$ has weight $j_1 + j_2 + \cdots + j_r$ and

$$\mathfrak{B} = \sum_{k=0}^{\infty} \mathfrak{B}_k\,, \tag{1}$$

where $\mathfrak{B}_k$ is the additive group of those polynomials which contain only terms of weight k and $\mathfrak{B}_0 = B$. The group $\mathfrak{B}_k$ is a module over B whose rank is equal to the number $\pi(k)$ of partitions of k. Clearly

$$\mathfrak{B}_r \mathfrak{B}_s \subset \mathfrak{B}_{r+s}\,. \tag{2}$$

1.2. Let $\{K_j\}$ be a sequence of polynomials in the indeterminates p_i with $K_0 = 1$ and $K_j \in \mathfrak{B}_j$ $(j = 0, 1, 2, \ldots)$. The sequence $\{K_j\}$ is called a *multiplicative sequence* (or m-sequence) if every identity of the form

$$\begin{aligned} &1 + p_1 z + p_2 z^2 + \cdots \\ &\qquad = (1 + p_1' z + p_2' z^2 + \cdots)(1 + p_1'' z + p_2'' z^2 + \cdots) \end{aligned} \tag{3}$$

with z, p_i', p_i'' indeterminate implies an identity

$$\begin{aligned} &\sum_{j=0}^{\infty} K_j(p_1, p_2, \ldots, p_j)\, z^j \\ &= \sum_{i=0}^{\infty} K_i(p_1', p_2', \ldots, p_i')\, z^i \sum_{j=0}^{\infty} K_j(p_1'', p_2'', \ldots, p_j'')\, z^j\,. \end{aligned} \tag{4}$$

In abbreviated notation we write

$$K\left(\sum_{j=0}^{\infty} p_i z^i\right) = \sum_{j=0}^{\infty} K_j(p_1, \ldots, p_j)\, z^j$$

both when the p_i are indeterminates and when they are replaced by particular values. The power series

$$K(1+z) = \sum_{i=0}^{\infty} b_i z^i \quad (b_0 = 1,\ b_i = K_i(1, 0, \ldots, 0) \in B)$$

is called the *characteristic power series* of the m-sequence $\{K_j\}$.

In the sequel we consider formal factorisations

$$1 + p_1 z + \cdots + p_m z^m = \prod_{i=1}^{m} (1 + \beta_i z)\,. \tag{5_m}$$

That is, the elements p_i are regarded as the elementary symmetric functions in $\beta_1, \ldots, \beta_m$. The ring $\mathfrak{B}$ is then the ring of all symmetric