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Peter J. Olver

Applications of Lie Groups to Differential Equations

Second Edition

李群在微分方程中的应用

第2版

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Preface to the First Edition

This book is devoted to explaining a wide range of applications of continuous symmetry groups to physically important systems of differential equations. Emphasis is placed on significant applications of group-theoretic methods, organized so that the applied reader can readily learn the basic computational techniques required for genuine physical problems. The first chapter collects together (but does not prove) those aspects of Lie group theory which are of importance to differential equations. Applications covered in the body of the book include calculation of symmetry groups of differential equations, integration of ordinary differential equations, including special techniques for Euler–Lagrange equations or Hamiltonian systems, differential invariants and construction of equations with prescribed symmetry groups, group-invariant solutions of partial differential equations, dimensional analysis, and the connections between conservation laws and symmetry groups. Generalizations of the basic symmetry group concept, and applications to conservation laws, integrability conditions, completely integrable systems and soliton equations, and bi-Hamiltonian systems are covered in detail. The exposition is reasonably self-contained, and supplemented by numerous examples of direct physical importance, chosen from classical mechanics, fluid mechanics, elasticity and other applied areas. Besides the basic theory of manifolds, Lie groups and algebras, transformation groups and differential forms, the book delves into the more theoretical subjects of prolongation theory and differential equations, the Cauchy–Kovalevskaya theorem, characteristics and integrability of differential equations, extended jet spaces over manifolds, quotient manifolds, adjoint and co-adjoint representations of Lie groups, the calculus of variations and the inverse problem of characterizing those systems which are Euler–Lagrange equations of some variational problem, differential operators, higher Euler operators and the variational complex, and the general theory of Poisson structures, both for finite-dimensional Hamiltonian systems as well as systems of evolution equations, all of which have direct bearing on the symmetry analysis of differential equations. It is hoped that after reading this book, the reader will, with a minimum of difficulty, be able to readily apply these important group-theoretic methods to the systems of differential equations he or she is interested in, and make new and interesting deductions concerning them. If so, the book can be said to have served its purpose.

A preliminary version of this book first appeared as a set of lecture notes, distributed by the Mathematical Institute of Oxford University, for a graduate seminar held in Trinity term, 1979. It is my pleasure to thank the staff of Springer-Verlag for their encouragement for me to turn these notes into book form, and for their patience during the process of revision that turned out to be far more extensive than I originally anticipated.

Preface to the Second Edition

For the second edition, I have corrected a number of misprints and inadvertent mathematical errors that found their way into the original version. More substantial changes are the inclusion of a simpler proof of Theorem 4.26 due to Alonso, [1], and the omission of the false (at least in the form stated in the first edition) Theorem 5.22 on the commutativity of generalized symmetries. Also, I have corrected some of the exercises and added several new ones. Hopefully this now eliminates all of the major (and almost all of the minor) mistakes. The one substantial addition to the second edition is a short presentation of the calculus of pseudo-differential operators and their use in Shabat's theory of formal symmetries, which provides a powerful, algorithmic method for determining the integrability of evolution equations.

The years since the appearance of the original edition of the book have witnessed a remarkable explosion of research, both pure and applied, into symmetry group methods in differential equations, proceeding at a pace well beyond my expectations. Innumerable papers, as well as several substantial textbooks devoted to the subject of symmetry and differential equations, have appeared in the literature. The former are too numerous to try to list here, although I have added a few of the more notable contributions to the list of references and have correspondingly updated the historical notes at the end of each chapter. Of the latter, I recommend the books of Bluman and Kumei, [2], and Stephani [3], on symmetry methods, and Zharinov, [1], on the geometrical theory of differential equations. There has also been a lot of activity in the development of computer algebra (symbolic manipulation) computer programs to (partially) automate the determination of symmetry groups of differential equations. A good survey of the available codes, as of 1991, including a discussion of their strengths and weaknesses, can be found in the paper of Champagne, Hereman, and Winternitz, [1].

I would like to acknowledge, with gratitude, Ian Anderson, Ken Driessel, Darryl Holm, Niky Kamran, John Maddocks, Jerry Marsden, Sascha Mikhailov, and Alexei Shabat, who offered valuable comments and suggestions for improving the first edition. Finally, I should reiterate my thankfulness and love to my wife, Cheri, and children, Pari, Sheehan, and Noreen, for their continued, all-important love and support!

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It is unfortunately impossible to mention all those colleagues who have, in some way, influenced my mathematical career. However, the following people deserve an especial thanks for their direct roles in aiding and abetting the preparation of this book (needless to say, I accept full responsibility for what appears in it!).

Garrett Birkhoff—my thesis advisor, who first introduced me to the marvellous world of Lie groups and expertly guided my first faltering steps in the path of mathematical research.

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Introduction

When beginning students first encounter ordinary differential equations they are, more often than not, presented with a bewildering variety of special techniques designed to solve certain particular, seemingly unrelated types of equations, such as separable, homogeneous or exact equations. Indeed, this was the state of the art around the middle of the nineteenth century, when Sophus Lie made the profound and far-reaching discovery that these special methods were, in fact, all special cases of a general integration procedure based on the invariance of the differential equation under a continuous group of symmetries. This observation at once unified and significantly extended the available integration techniques, and inspired Lie to devote the remainder of his mathematical career to the development and application of his monumental theory of continuous groups. These groups, now universally known as Lie groups, have had a profound impact on all areas of mathematics, both pure and applied, as well as physics, engineering and other mathematically-based sciences. The applications of Lie's continuous symmetry groups include such diverse fields as algebraic topology, differential geometry, invariant theory, bifurcation theory, special functions, numerical analysis, control theory, classical mechanics, quantum mechanics, relativity, continuum mechanics and so on. It is impossible to overestimate the importance of Lie's contribution to modern science and mathematics.

Nevertheless, anyone who is already familiar with one of these modern manifestations of Lie group theory is perhaps surprised to learn that its original inspirational source was the field of differential equations. One possible cause for the general lack of familiarity with this important aspect of Lie group theory is the fact that, as with many applied fields, the Lie groups that do arise as symmetry groups of genuine physical systems of differential equations are often not particularly elegant groups from a purely mathematical viewpoint, being neither semi-simple, nor solvable, nor any of the other special classes of Lie groups so popular in mathematics. Moreover, these groups often act nonlinearly on the underlying space (taking us outside the domain of representation theory) and can even be only locally defined, with the transformations making sense only for group elements sufficiently near the identity. The relevant group actions, then, are much closer in spirit to Lie's original formulation of the subject in terms of local Lie groups acting on open subsets of Euclidean space, and runs directly counter to the modern tendencies towards abstraction and globalization which have enveloped much of present-day Lie group theory. Historically, the applications of Lie groups to differential equations pioneered by Lie and Noether faded into obscurity just as the global, abstract reformulation of differential geometry and Lie group theory championed by E. Cartan gained its ascendency in the mathematical community. The entire subject lay dormant for nearly half a century until G. Birkhoff called attention to the unexploited applications of Lie groups to the differential equations of fluid mechanics. Subsequently, Ovsiannikov and his school began a systematic program of successfully applying these methods to a wide range of physically important problems. The last two decades have witnessed a veritable explosion of research activity in this field, both in the applications to concrete physical systems, as well as extensions of the scope and depth of the theory itself. Nevertheless, many questions remain unresolved, and the full range of applicability of Lie group methods to differential equations is yet to be determined.

Roughly speaking, a symmetry group of a system of differential equations is a group which transforms solutions of the system to other solutions. In the classical framework of Lie, these groups consist of geometric transformations on the space of independent and dependent variables for the system, and act on solutions by transforming their graphs. Typical examples are groups of translations and rotations, as well as groups of scaling symmetries, but these certainly do not exhaust the range of possibilities. The great advantage of looking at continuous symmetry groups, as opposed to discrete symmetries such as reflections, is that they can all be found using explicit computational methods. This is not to say that discrete groups are not important in the study of differential equations (see, for example, Hejhal, [1], and the references therein), but rather that one must employ quite different methods to find or utilize them. Lie's fundamental discovery was that the complicated nonlinear conditions of invariance of the system under the group transformations could, in the case of a continuous group, be replaced by equivalent, but far simpler, *linear* conditions reflecting a form of "infinitesimal" invariance of the system under the generators of the group. In almost every physically important system of differential equations, these infinitesimal symmetry conditions—the so-called defining equations of the symmetry group of the system—can be explicitly solved in closed form and thus the most general continuous symmetry group of the system can be explicitly determined. The entire procedure consists of rather mechanical computations, and, indeed, several symbolic manipulation computer programs have been developed for this task.

Once one has determined the symmetry group of a system of differential equations, a number of applications become available. To start with, one can directly use the defining property of such a group and construct new solutions to the system from known ones. The symmetry group thus provides a means of classifying different symmetry classes of solutions, where two solutions are deemed to be equivalent if one can be transformed into the other by some group element. Alternatively, one can use the symmetry groups to effect a classification of families of differential equations depending on arbitrary parameters or functions; often there are good physical or mathematical reasons for preferring those equations with as high a degree of symmetry as possible. Another approach is to determine which types of differential equations admit a prescribed group of symmetries; this problem is also answered by infinitesimal methods using the theory of differential invariants.

In the case of ordinary differential equations, invariance under a one-parameter symmetry group implies that we can reduce the order of the equation by one, recovering the solutions to the original equation from those of the reduced equation by a single quadrature. For a single first order equation, this method provides an explicit formula for the general solution. Multi-parameter symmetry groups engender further reductions in order, but, unless the group itself satisfies an additional "solvability" requirement, we may not be able to recover the solutions to the original equation from those of the reduced equation by quadratures alone. If the system of ordinary differential equations is derived from a variational principle, either as the Euler–Lagrange equations of some functional, or as a Hamiltonian system, then the power of the symmetry group reduction method is effectively doubled. A one-parameter group of "variational" symmetries allows one to reduce the order of the system by two; the case of multi-parameter symmetry groups is more delicate.

Unfortunately, for systems of partial differential equations, the symmetry group is usually of no help in determining the general solution (although in special cases it may indicate when the system can be transformed into a more easily soluble system such as a linear system). However, one can use general symmetry groups to explicitly determine special types of solutions which are themselves invariant under some subgroup of the full symmetry group of the system. These "group-invariant" solutions are found by solving a reduced system of differential equations involving fewer independent variables than the original system (which presumably makes it easier to solve). For example, the solutions to a partial differential equation in two independent variables which are invariant under a given one-parameter symmetry group are all found by solving a system of ordinary differential equations. Included among these general group-invariant solutions are the classical similarity solutions coming from groups of scaling symmetries, and travelling wave solutions reflecting some form of translational invariance in the system, as well as many other explicit solutions of direct physical or mathematical importance. For many nonlinear systems, these are the *only* explicit, exact solutions which are available, and, as such, play an important role in both the mathematical analysis and physical applications of the system.

In 1918, E. Noether proved two remarkable theorems relating symmetry groups of a variational integral to properties of its associated Euler–Lagrange equations. In the first of these theorems, Noether shows how each one-parameter variational symmetry group gives rise to a conservation law of the Euler–Lagrange equations. Thus, for example, conservation of energy comes from the invariance of the problem under a group of time translations, while conservation of linear and angular momenta reflect translational and rotational invariance of the system. Chapter 4 is devoted to the so-called classical form of Noether's theorem, in which only the geometrical types of symmetry groups are used. Noether herself proved a far more general result and gave a one-to-one correspondence between symmetry groups and conservation laws. The general result necessitates the introduction of "generalized symmetries" which are groups whose infinitesimal generators depend not only on the independent and dependent variables of the system, but also the derivatives of the dependent variables. The corresponding group transformations will no longer act geometrically on the space of independent and dependent variables, transforming a function's graph point-wise, but are nonlocal transformations found by integrating an evolutionary system of partial differential equations. Each one-parameter group of symmetries of a variational problem, either geometrical or generalized, will give rise to a conservation law, and, conversely, every conservation law arises in this manner. The simplest example of a conserved quantity coming from a true generalized symmetry is the Runge–Lenz vector for the Kepler problem, but additional recent applications, including soliton equations and elasticity, has sparked a renewed interest in the general version of Noether's theorem. In Section 5.3 we prove a strengthened form of Noether's theorem, stating that for "normal" systems there is in fact a one-to-one correspondence between *nontrivial* variational symmetry groups and *nontrivial* conservation laws. The condition of normality is satisfied by most physically important systems of differential equations; abnormal systems are essentially those with nontrivial integrability conditions. An important class of abnormal systems, which do arise in general relativity, are those whose variational integral admits an infinite-dimensional symmetry group depending on an arbitrary function. Noether's second theorem shows that there is then a nontrivial relation among the ensuing Euler–Lagrange equations, and, consequently, nontrivial symmetries giving rise to only trivial conservation laws. Once found, conservation laws have many important applications, both physical and mathematical, including existence results, shock waves, scattering theory, stability, relativity, fluid mechanics, elasticity and so on. See the notes on Chapter 4 for a more extensive list, including references.

Neglected for many years following Noether's prescient work, generalized symmetries have recently been found to be of importance in the study of nonlinear partial differential equations which, like the Korteweg–de Vries equation, can be viewed as "completely integrable systems". The existence of infinitely many generalized symmetries, usually found via the recursion operator methods of Section 5.2, appears to be intimately connected with the possibility of linearizing the system, either directly through some change of variables, or, more subtly, through some form of inverse scattering method. Thus, the generalized symmetry approach, which is amenable to direct calculation as with ordinary symmetries, provides a systematic means of recognizing these remarkable equations and thereby constructing an infinite collection of conservation laws for them. (The construction of the related scattering problem requires different techniques such as the prolongation methods of Wahlquist and Estabrook, [1].) A systematic method for determining evolution equations having recursion operators, and hence classifying "integrable" systems, is provided by the theory of formal symmetries.

A number of the applications of symmetry group methods to partial differential equations are most naturally done using some form of Hamiltonian structure. The finite-dimensional formulation of Hamiltonian systems of ordinary differential equations is well known, but in preparation for the more recent theory of Hamiltonian systems of evolution equations, we are required to take a slightly novel approach to the whole subject of Hamiltonian mechanics. Here we will de-emphasize the use of canonical coordinates (the p's and q's of classical mechanics) and concentrate instead on the Poisson bracket as the cornerstone of the subject. The result is the more general concept of a Poisson structure, which *is* easily extended to include the infinite-dimensional theory of Hamiltonian systems of evolution equations. An important special case of a Poisson structure is the Lie–Poisson structure on the dual to a Lie algebra, originally discovered by Lie, and more recently used to great effect in geometric quantization, representation theory, and fluid and plasma mechanics. In this general approach to Hamiltonian mechanics, conservation laws can arise not only from symmetry properties of the system, but also from degeneracies of the Poisson bracket itself. In the finite-dimensional set-up, each one-parameter Hamiltonian symmetry group allows us to reduce the order of a system by two. In its modern formulation, the degree of reduction available for multi-parameter symmetry groups is given by the general theory of Marsden and Weinstein, which is based on the concept of a momentum map to the dual of the symmetry Lie algebra. In more recent work, there has been a fair amount of interest in systems of differential equations which possess not just one, but *two* distinct (but compatible) Hamiltonian structures. For such a "bi-Hamiltonian system", there is a direct recursive means of constructing an infinite hierarchy of mutually commuting flows (symmetries) and consequent conservation laws, indicating the system's complete integrability. Most of the soliton equations, as well as some of the finite-dimensional completely integrable Hamiltonian systems, are in fact bi-Hamiltonian systems.

Underlying much of the theory of generalized symmetries, conservation laws, and Hamiltonian structures for evolution equations is a subject known as the "formal calculus of variations", which constitutes a calculus specifically devised for answering a wide range of questions dealing with complicated algebraic identities among objects such as the Euler operator from the calculus of variations, generalized symmetries, total derivatives and more general differential operators, and several generalizations of the concept of a differential form. The principal result in the formal variational calculus is the local exactness of a certain complex—called the "variational complex"—which is in a sense the proper generalization to the variational or jet space context of the de Rham complex from algebraic topology. In recent years, this variational complex has been seen to play an increasingly important role in the development of the algebraic and geometric theory of the calculus of variations. Included as special cases of the variational complex are:

(1) a solution to the "inverse problem of the calculus of variations", which is to characterize those systems of differential equations which are the Euler–Lagrange equations for some variational problem;
(2) the characterization of "null Lagrangians", meaning those variational integrals whose Euler–Lagrange expressions vanish identically, as total divergences; and
(3) the characterization of trivial conservation laws, also known as "null divergences", as "total curls".

Each of these results is vital to the development of our applications of Lie groups to the study of conservation laws and Hamiltonian structures for evolution equations. Since it is not much more difficult to provide the proof of exactness of the full variational complex, Section 5.4 is devoted to a complete development of this proof and application to the three special cases of interest.

Although the book covers a wide range of different applications of Lie groups to differential equations, a number of important topics have necessarily been omitted. Most notable among these omissions is the connection between Lie groups and separation of variables. There are two reasons for this: first, there is an excellent, comprehensive text—Miller, [3]—already available; second, except for special classes of partial differential equations, such as Hamilton–Jacobi and Helmholtz equations, the precise connections between symmetries and separation of variables is not well understood at present. This is especially true in the case of systems of linear equations, or for fully nonlinear separation of variables; in neither case is there even a good definition of what separation of variables really entails, let alone how one uses symmetry properties of the system to detect coordinate systems in which separation of variables is possible. I have also not attempted to cover any of the vast area of representation theory, and the consequent applications to special function theory; see Miller, [1] or Vilenkin, [1]. Bifurcation theory is another fertile ground for group-theoretic applications; I refer the reader to the lecture notes of Sattinger, [1], and the references therein. Applications of symmetry groups to numerical analysis are given extensive treatment in Shokin, [1], and Dorodnitsyn, [1]. Applications to control theory can be found in van der Schaft, [1], and Ramakrishnan and Schaettler, [1]. See Maeda, [1], and Levi and Winternitz, [2], for applications to difference and differential-difference equations. Extensions of the present methods to boundary value problems for partial differential equations can be found in the books of Bluman and Cole, [1], and Seshadri and Na, [1], and to free boundary problems in Benjamin and Olver, [1]. Although I have given an extensive treatment to generalized symmetries in Chapter 5, the related concept of contact transformations introduced by Lie has not been covered, as it seems much less relevant to the equations arising in applications, and, for the most part, is subsumed by the more general theory presented here; see Anderson and Ibragimov, [1], Bluman and Kumei, [2], and the references therein for these types of transformation groups. Finally, we should mention the use of Lie group methods for differential equations arising in geometry, including, for example, motions in Riemannian manifolds, cf. Ibragimov, [1], or symmetric spaces and invariant differential operators associated with them, cf. Helgason, [1], [2].

Notes to the Reader

The guiding principle in the organization of this book has been so as to enable the reader whose principal goal is to apply Lie group methods to concrete problems to learn the basic computational tools and techniques as quickly as possible and with a minimum of theoretical diversions. At the same time, the computational applications have been placed on a solid theoretical foundation, so that the more mathematically inclined reader can readily delve further into the subject. Each chapter following the first has been arranged so that the applications and examples appear as quickly as feasible, with the more theoretical proofs and explanations coming towards the end. Even should the reader have more theoretical goals in mind, though, I would still strongly recommend that they learn the computational techniques and examples first before proceeding to the general theory. It has been said that it is far easier to abstract a general mathematical theory from a single well-chosen example than it is to apply an existing abstract theory to a specific example, and this, I believe, is certainly the case here.

For the reader whose main interest is in applications, I would recommend the following strategy for reading the book. The principal question is how much of the introductory theory of manifolds, vector fields, Lie groups and Lie algebras (which has, for convenience, been collected together in Chapter 1 and Section 2.1), really needs to be covered before one can proceed to the applications to differential equations starting in Section 2.2. The answer is, in fact, surprisingly little. Manifolds can for the most part be thought of locally, as open subsets of a Euclidean space $\mathbb{R}^m$ in which one has the freedom to change coordinates as one desires. Geometrical symmetry groups will just be collections of transformations on such a subset which satisfy certain elementary group axioms allowing one to compose successive symmetries, take inverses, etc. The key concept in the subject is the infinitesimal generator of a