Wiley Series in Operations Research and Management Science

# ELEMENTS OF RANDOM WALK AND DIFFUSION PROCESSES

Oliver C. Ibe

WILEY

# ELEMENTS OF RANDOM WALK AND DIFFUSION PROCESSES

**OLIVER C. IBE**

University of Massachusetts
Lowell, Massachusetts



WILEY

Published by John Wiley & Sons, Inc., Hoboken, New Jersey
Published simultaneously in Canada



***Library of Congress Cataloging-in-Publication Data:***

Ibe, Oliver C. (Oliver Chukwudi), 1947–
Elements of random walk and diffusion processes / Oliver C. Ibe, University of Massachusetts Lowell, MA.
pages cm
Includes bibliographical references and index.
ISBN 978-1-118-61809-7 (hardback)
1. Random walks (Mathematics) 2. Diffusion processes. I. Title.
QA274.73.I24 2013
519.2′82–dc23

2013009389

Printed in the United States of America

ISBN: 9781118618097

10 9 8 7 6 5 4 3 2 1

# ELEMENTS OF RANDOM WALK AND DIFFUSION PROCESSES

Wiley Series in

# Operations Research and Management Science

Operations Research and Management Science (ORMS) is a broad, interdisciplinary branch of applied mathematics concerned with improving the quality of decisions and processes and is a major component of the global modern movement towards the use of advanced analytics in industry and scientific research. The *Wiley Series in Operations Research and Management Science* features a broad collection of books that meet the varied needs of researchers, practitioners, policy makers, and students who use or need to improve their use of analytics. Reflecting the wide range of current research within the ORMS community, the Series encompasses application, methodology, and theory and provides coverage of both classical and cutting edge ORMS concepts and developments. Written by recognized international experts in the field, this collection is appropriate for students as well as professionals from private and public sectors including industry, government, and nonprofit organization who are interested in ORMS at a technical level. The Series is comprised of three sections: Decision and Risk Analysis; Optimization Models; and Stochastic Models.



---

**Decision and Risk Analysis**

Barron • *Game Theory: An Introduction,* Second Edition

*Forthcoming Titles*

Nussbaum and Mislick • *Cost Estimation: Methods and Tools Optimization Models*

**Optimization Models**

Ghiani, Laporte, and Musmanno • *Introduction to Logistics Systems Management,* Second Edition

**Stochastic Models**

Ibe • *Elements of Random Walk and Diffusion Processes*

# PREFACE

This book is about random walks and the related subject of diffusion processes. The simplest definition of a random walk is that it is a stochastic process that consists of a sequence of discrete steps of fixed length. A more rigorous mathematical definition of a random walk is that it is a stochastic process that is formed by successive summation of independent and identically distributed random variables. In other words, it is a mathematical formalization of a trajectory that consists of taking successive random steps.

Random walk is related to numerous physical processes, including the Brownian motion, diffusion processes, and Levy flights. Consequently, it is used to explain observed behaviors of processes in several fields, including ecology, economics, psychology, computer science, physics, chemistry, and biology. The path traced by a molecule as it travels in a liquid or a gas, the search path of a foraging animal, the price of a fluctuating stock, and the financial status of a gambler can all be modeled as random walks. For example, the trajectory of insects on a horizontal planar surface may be accurately modeled as a random walk. In polymer physics, random walk is the simplest model to study polymers. Random walk accurately explains the relation between the time needed to make a decision and the probability that a certain decision will be made, which is a concept that is used in psychology. It is used in wireless networking to model node movements and node failures. It has been used in computer science to estimate the size of the World Wide Web. Also, in image processing, the random walker segmentation algorithm is used to determine the labels to associate with each pixel. Thus, random walk is a stochastic process that has proven to be a useful model in understanding several processes across a wide spectrum of scientific disciplines.

There are different types of random walks. Some random walks are on graphs while others are on the line, in the plane, or in higher dimensions. Random walks also vary with regard to the time parameter. In discrete-time random walks, the walker usually takes fixed-length steps in discrete time. In continuous-time random walks, the walker takes steps at random times, and the step length is usually a random variable.

The purpose of this book is to bring into one volume the different types of random walks and related topics, including Brownian motion, diffusion processes, and Levy flights. While many books have been written on random walks, our approach in this book is different. The book includes standard methods and applications of Brownian motion, which is considered a limit of random walk, and a discussion on Levy flights and Levy walk, which have become popular in random searches in ecology, finance, and other fields. It also includes a chapter on fractional calculus that is necessary for understanding anomalous (or fractional) diffusion, fractional Brownian motion, and fractional random walk. Finally, it introduces the reader to percolation theory and its relationship to diffusion processes. It is a self-contained book that will appeal to graduate students across science, engineering, and mathematics who need to understand the applications of random walk and diffusion process techniques, as well as to established researchers. It presents the connections between diffusion equations and random walks and introduces stochastic calculus, which is a prerequisite for understanding some of the modern concepts in Brownian motion and diffusion processes.

The chapters are organized as follows. Chapter 1 presents an introduction to probability while Chapter 2 gives an overview of stochastic processes. Chapter 3 discusses one-dimensional random walk while Chapter 4 discusses two-dimensional random walk. Chapter 5 discusses Brownian motion, Chapter 6 presents an introduction to stochastic calculus, and Chapter 7 discusses diffusion processes. Chapter 8 discusses Levy flights and Levy walk, Chapter 9 discusses fractional calculus and its applications, and Chapter 10 discusses percolation theory.

The book is written with the understanding that much of research on social networks, economics, finance, ecology, biostatistics, polymer physics, and population genetics has become interdisciplinary with random walk and diffusion processes as the common thread. Thus, it is truly a book designed for interdisciplinary use. It can be used for a one-semester course on stochastic processes and their applications.

OLIVER C. IBE

# ACKNOWLEDGMENTS

My journey into the field of stochastic systems modeling started with my encounter at the Massachusetts Institute of Technology with two giants in the field, namely, the late Professor Alvin Drake, who was my academic adviser and under whom I was a teaching assistant in a course titled "Probabilistic Systems Analysis," and Professor Robert Gallager, who was my adviser for both my master's and doctoral theses. This book is a product of the wonderful instruction I received from these two great professors, and I sincerely appreciate all that they did to get me excited in this field.

This is the second project that I have completed with my editor, Ms. Susanne Steitz-Filler of Wiley. I am sincerely grateful to her for encouraging me and for being patient with me throughout the time it took to get the project completed. I would also like to thank Ms. Sari Friedman, an Editorial Assistant at Wiley, for ensuring that the production schedule is met. This is also my second project with her. I am grateful to the anonymous reviewers for their comments that helped to improve the quality of the book.

Finally, I would like to thank my wife, Christina, and our children Chidinma, Ogechi, Amanze, and Ugonna, for the joy they have brought to my life. I could not have completed this project without their encouragement.

# CONTENTS