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# **ELEMENTS OF RANDOM WALK AND DIFFUSION PROCESSES**

**Oliver C. Ibe**

**WILEY**

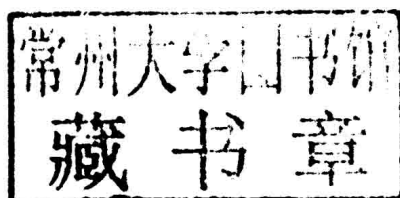
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# ELEMENTS OF RANDOM WALK AND DIFFUSION PROCESSES

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# PREFACE

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This book is about random walks and the related subject of diffusion processes. The simplest definition of a random walk is that it is a stochastic process that consists of a sequence of discrete steps of fixed length. A more rigorous mathematical definition of a random walk is that it is a stochastic process that is formed by successive summation of independent and identically distributed random variables. In other words, it is a mathematical formalization of a trajectory that consists of taking successive random steps.

Random walk is related to numerous physical processes, including the Brownian motion, diffusion processes, and Levy flights. Consequently, it is used to explain observed behaviors of processes in several fields, including ecology, economics, psychology, computer science, physics, chemistry, and biology. The path traced by a molecule as it travels in a liquid or a gas, the search path of a foraging animal, the price of a fluctuating stock, and the financial status of a gambler can all be modeled as random walks. For example, the trajectory of insects on a horizontal planar surface may be accurately modeled as a random walk. In polymer physics, random walk is the simplest model to study polymers. Random walk accurately explains the relation between the time needed to make a decision and the probability that a certain decision will be made, which is a concept that is used in psychology. It is used in wireless networking to model node movements and node failures. It has been used in computer science to estimate the size of the World Wide Web. Also, in image processing, the random walker segmentation algorithm is used to determine the labels to associate with each pixel. Thus, random walk is a stochastic process that has proven to be a useful model in understanding several processes across a wide spectrum of scientific disciplines.

There are different types of random walks. Some random walks are on graphs while others are on the line, in the plane, or in higher dimensions. Random walks also

vary with regard to the time parameter. In discrete-time random walks, the walker usually takes fixed-length steps in discrete time. In continuous-time random walks, the walker takes steps at random times, and the step length is usually a random variable.

The purpose of this book is to bring into one volume the different types of random walks and related topics, including Brownian motion, diffusion processes, and Levy flights. While many books have been written on random walks, our approach in this book is different. The book includes standard methods and applications of Brownian motion, which is considered a limit of random walk, and a discussion on Levy flights and Levy walk, which have become popular in random searches in ecology, finance, and other fields. It also includes a chapter on fractional calculus that is necessary for understanding anomalous (or fractional) diffusion, fractional Brownian motion, and fractional random walk. Finally, it introduces the reader to percolation theory and its relationship to diffusion processes. It is a self-contained book that will appeal to graduate students across science, engineering, and mathematics who need to understand the applications of random walk and diffusion process techniques, as well as to established researchers. It presents the connections between diffusion equations and random walks and introduces stochastic calculus, which is a prerequisite for understanding some of the modern concepts in Brownian motion and diffusion processes.

The chapters are organized as follows. Chapter 1 presents an introduction to probability while Chapter 2 gives an overview of stochastic processes. Chapter 3 discusses one-dimensional random walk while Chapter 4 discusses two-dimensional random walk. Chapter 5 discusses Brownian motion, Chapter 6 presents an introduction to stochastic calculus, and Chapter 7 discusses diffusion processes. Chapter 8 discusses Levy flights and Levy walk, Chapter 9 discusses fractional calculus and its applications, and Chapter 10 discusses percolation theory.

The book is written with the understanding that much of research on social networks, economics, finance, ecology, biostatistics, polymer physics, and population genetics has become interdisciplinary with random walk and diffusion processes as the common thread. Thus, it is truly a book designed for interdisciplinary use. It can be used for a one-semester course on stochastic processes and their applications.

OLIVER C. IBE

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My journey into the field of stochastic systems modeling started with my encounter at the Massachusetts Institute of Technology with two giants in the field, namely, the late Professor Alvin Drake, who was my academic adviser and under whom I was a teaching assistant in a course titled “Probabilistic Systems Analysis,” and Professor Robert Gallager, who was my adviser for both my master’s and doctoral theses. This book is a product of the wonderful instruction I received from these two great professors, and I sincerely appreciate all that they did to get me excited in this field.

This is the second project that I have completed with my editor, Ms. Susanne Steitz-Filler of Wiley. I am sincerely grateful to her for encouraging me and for being patient with me throughout the time it took to get the project completed. I would also like to thank Ms. Sari Friedman, an Editorial Assistant at Wiley, for ensuring that the production schedule is met. This is also my second project with her. I am grateful to the anonymous reviewers for their comments that helped to improve the quality of the book.

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# CONTENTS

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<b>PREFACE</b>	<b>xiii</b>
<b>ACKNOWLEDGMENTS</b>	<b>xv</b>
<b>1 REVIEW OF PROBABILITY THEORY</b>	<b>1</b>
1.1 Introduction	1
1.2 Random Variables	1
1.2.1 Distribution Functions	2
1.2.2 Discrete Random Variables	3
1.2.3 Continuous Random Variables	3
1.2.4 Expectations	4
1.2.5 Moments of Random Variables and the Variance	4
1.3 Transform Methods	5
1.3.1 The Characteristic Function	5
1.3.2 Moment-Generating Property of the Characteristic Function	6
1.3.3 The $s$ -Transform	6
1.3.4 Moment-Generating Property of the $s$ -Transform	7
1.3.5 The $z$ -Transform	7
1.3.6 Moment-Generating Property of the $z$ -Transform	8
1.4 Covariance and Correlation Coefficient	9
	<b>v</b>

1.5	Sums of Independent Random Variables	10
1.6	Some Probability Distributions	11
1.6.1	The Bernoulli Distribution	11
1.6.2	The Binomial Distribution	12
1.6.3	The Geometric Distribution	12
1.6.4	The Poisson Distribution	13
1.6.5	The Exponential Distribution	13
1.6.6	Normal Distribution	14
1.7	Limit Theorems	16
1.7.1	Markov Inequality	16
1.7.2	Chebyshev Inequality	17
1.7.3	Laws of Large Numbers	17
1.7.4	The Central Limit Theorem	18
	Problems	19
<b>2</b>	<b>OVERVIEW OF STOCHASTIC PROCESSES</b>	<b>21</b>
2.1	Introduction	21
2.2	Classification of Stochastic Processes	22
2.3	Mean and Autocorrelation Function	22
2.4	Stationary Processes	23
2.4.1	Strict-Sense Stationary Processes	23
2.4.2	Wide-Sense Stationary Processes	24
2.5	Power Spectral Density	24
2.6	Counting Processes	25
2.7	Independent Increment Processes	25
2.8	Stationary Increment Process	25
2.9	Poisson Processes	26
2.9.1	Compound Poisson Process	28
2.10	Markov Processes	29
2.10.1	Discrete-Time Markov Chains	30
2.10.2	State Transition Probability Matrix	31
2.10.3	The $k$ -Step State Transition Probability	31
2.10.4	State Transition Diagrams	32
2.10.5	Classification of States	33
2.10.6	Limiting-State Probabilities	34
2.10.7	Doubly Stochastic Matrix	35
2.10.8	Continuous-Time Markov Chains	35
2.10.9	Birth and Death Processes	36

2.11	Gaussian Processes	38
2.12	Martingales	38
2.12.1	Stopping Times	40
	Problems	41

### **3 ONE-DIMENSIONAL RANDOM WALK 44**

3.1	Introduction	44
3.2	Occupancy Probability	46
3.3	Random Walk as a Markov Chain	49
3.4	Symmetric Random Walk as a Martingale	49
3.5	Random Walk with Barriers	50
3.6	Mean-Square Displacement	50
3.7	Gambler's Ruin	52
3.7.1	Ruin Probability	52
3.7.2	Alternative Derivation of Ruin Probability	54
3.7.3	Duration of a Game	55
3.8	Random Walk with Stay	56
3.9	First Return to the Origin	57
3.10	First Passage Times for Symmetric Random Walk	59
3.10.1	First Passage Time via the Generating Function	59
3.10.2	First Passage Time via the Reflection Principle	61
3.10.3	Hitting Time and the Reflection Principle	64
3.11	The Ballot Problem and the Reflection Principle	65
3.11.1	The Conditional Probability Method	66
3.12	Returns to the Origin and the Arc-Sine Law	67
3.13	Maximum of a Random Walk	72
3.14	Two Symmetric Random Walkers	73
3.15	Random Walk on a Graph	73
3.15.1	Proximity Measures	75
3.15.2	Directed Graphs	75
3.15.3	Random Walk on an Undirected Graph	76
3.15.4	Random Walk on a Weighted Graph	80
3.16	Random Walks and Electric Networks	80
3.16.1	Harmonic Functions	82
3.16.2	Effective Resistance and Escape Probability	82
3.17	Correlated Random Walk	85
3.18	Continuous-Time Random Walk	90
3.18.1	The Master Equation	92

3.19	Reinforced Random Walk	94
3.19.1	Polya's Urn Model	94
3.19.2	ERRW and Polya's Urn	96
3.19.3	ERRW Revisited	97
3.20	Miscellaneous Random Walk Models	98
3.20.1	Geometric Random Walk	98
3.20.2	Gaussian Random Walk	99
3.20.3	Random Walk with Memory	99
3.21	Summary	100
	Problems	100
<b>4</b>	<b>TWO-DIMENSIONAL RANDOM WALK</b>	<b>103</b>
4.1	Introduction	103
4.2	The Pearson Random Walk	105
4.2.1	Mean-Square Displacement	105
4.2.2	Probability Distribution	107
4.3	The Symmetric 2D Random Walk	110
4.3.1	Stirling's Approximation of Symmetric Walk	112
4.3.2	Probability of Eventual Return for Symmetric Walk	113
4.3.3	Mean-Square Displacement	114
4.3.4	Two Independent Symmetric 2D Random Walkers	114
4.4	The Alternating Random Walk	115
4.4.1	Stirling's Approximation of Alternating Walk	117
4.4.2	Probability of Eventual Return for Alternating Walk	117
4.5	Self-Avoiding Random Walk	117
4.6	Nonreversing Random Walk	121
4.7	Extensions of the NRRW	126
4.7.1	The Noncontinuing Random Walk	126
4.7.2	The Nonreversing and Noncontinuing Random Walk	127
4.8	Summary	128
<b>5</b>	<b>BROWNIAN MOTION</b>	<b>129</b>
5.1	Introduction	129
5.2	Brownian Motion with Drift	132

5.3	Brownian Motion as a Markov Process	132
5.4	Brownian Motion as a Martingale	133
5.5	First Passage Time of a Brownian Motion	133
5.6	Maximum of a Brownian Motion	135
5.7	First Passage Time in an Interval	135
5.8	The Brownian Bridge	136
5.9	Geometric Brownian Motion	137
5.10	The Langevin Equation	137
5.11	Summary	141
	Problems	141

## **6 INTRODUCTION TO STOCHASTIC CALCULUS 143**

6.1	Introduction	143
6.2	The Ito Integral	145
6.3	The Stochastic Differential	146
6.4	The Ito's Formula	147
6.5	Stochastic Differential Equations	147
6.6	Solution of the Geometric Brownian Motion	148
6.7	The Ornstein–Uhlenbeck Process	151
6.7.1	Solution of the Ornstein–Uhlenbeck SDE	152
6.7.2	First Alternative Solution Method	153
6.7.3	Second Alternative Solution Method	154
6.8	Mean-Reverting Ornstein–Uhlenbeck Process	155
6.9	Summary	157

## **7 DIFFUSION PROCESSES 158**

7.1	Introduction	158
7.2	Mathematical Preliminaries	159
7.3	Diffusion on One-Dimensional Random Walk	160
7.3.1	Alternative Derivation	163
7.4	Examples of Diffusion Processes	164
7.4.1	Brownian Motion	164
7.4.2	Brownian Motion with Drift	167
7.5	Correlated Random Walk and the Telegraph Equation	167
7.6	Diffusion at Finite Speed	170
7.7	Diffusion on Symmetric Two-Dimensional Lattice Random Walk	171
7.8	Diffusion Approximation of the Pearson Random Walk	173
7.9	Summary	174

<b>8</b>	<b>LEVY WALK</b>	<b>175</b>
8.1	Introduction	175
8.2	Generalized Central Limit Theorem	175
8.3	Stable Distribution	177
8.4	Self-Similarity	182
8.5	Fractals	183
8.6	Levy Distribution	185
8.7	Levy Process	186
8.8	Infinite Divisibility	186
8.8.1	The Infinite Divisibility of the Poisson Process	187
8.8.2	Infinite Divisibility of the Compound Poisson Process	187
8.8.3	Infinite Divisibility of the Brownian Motion with Drift	188
8.9	Levy Flight	188
8.9.1	First Passage Time of Levy Flights	190
8.9.2	Leapover Properties of Levy Flights	190
8.10	Truncated Levy Flight	191
8.11	Levy Walk	191
8.11.1	Levy Walk as a Coupled CTRW	192
8.11.2	Truncated Levy Walk	195
8.12	Summary	195
<b>9</b>	<b>FRACTIONAL CALCULUS AND ITS APPLICATIONS</b>	<b>196</b>
9.1	Introduction	196
9.2	Gamma Function	197
9.3	Mittag–Leffler Functions	198
9.4	Laplace Transform	200
9.5	Fractional Derivatives	202
9.6	Fractional Integrals	203
9.7	Definitions of Fractional Integro-Differentials	203
9.7.1	Riemann–Liouville Fractional Derivative	204
9.7.2	Caputo Fractional Derivative	205
9.7.3	Grunwald–Letnikov Fractional Derivative	206
9.8	Fractional Differential Equations	207
9.8.1	Relaxation Differential Equation of Integer Order	208
9.8.2	Oscillation Differential Equation of Integer Order	208
9.8.3	Relaxation and Oscillation Fractional Differential Equations	209

9.9	Applications of Fractional Calculus	210
9.9.1	Fractional Brownian Motion	210
9.9.2	Multifractional Brownian Motion	213
9.9.3	Fractional Random Walk	213
9.9.4	Fractional (or Anomalous) Diffusion	215
9.9.5	Fractional Gaussian Noise	221
9.9.6	Fractional Poisson Process	222
9.10	Summary	224
<b>10</b>	<b>PERCOLATION THEORY</b>	<b>225</b>
10.1	Introduction	225
10.2	Graph Theory Revisited	226
10.2.1	Complete Graphs	226
10.2.2	Random Graphs	226
10.3	Percolation on a Lattice	228
10.3.1	Cluster Formation and Phase Transition	229
10.3.2	Percolation Probability and Critical Exponents	233
10.4	Continuum Percolation	235
10.4.1	The Boolean Model	235
10.4.2	The Random Connection Model	236
10.5	Bootstrap (or $k$ -Core) Percolation	237
10.6	Diffusion Percolation	237
10.6.1	Bootstrap Percolation versus Diffusion Percolation	239
10.7	First-Passage Percolation	239
10.8	Explosive Percolation	240
10.9	Percolation in Complex Networks	242
10.9.1	Average Path Length	243
10.9.2	Clustering Coefficient	243
10.9.3	Degree Distribution	244
10.9.4	Percolation and Network Resilience	244
10.10	Summary	245
<b>REFERENCES</b>		<b>247</b>
<b>INDEX</b>		<b>253</b>

