International Series on Applied Mathematics

POLYNOMIALS ORTHOGONAL ON A CIRCLE AND INTERVAL

by

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PERGAMON PRESS
OXFORD · LONDON · NEW YORK · PARIS

1960

INTRODUCTION

THE theory of orthogonal polynomials is of constant interest to mathematicians and physicists because, after the simplest orthogonal system - the trigonometrical - the system of orthogonal polynomials is the most simple.

One of the most interesting, and from the point of view of application, the most important, problems of this theory is that of the conditions of convergence of the expansion of a given function into a series of orthogonal polynomials. To resolve this problem we must know the asymptotic properties of orthogonal polynomials, i.e., their behaviour when the order increases indefinitely. The asymptotic formulae for orthogonal polynomials in the general case were found by G.Szëgo [5] and S.N.Bernstein [1]*.

The first task of the present monograph is to establish the proof of the asymptotic formulae under more general conditions than determined by previous authors (in particular, under certain local conditions).

In many cases, however, the existence of an asymptotic formula is not as important as the boundedness of the orthogonal system on the entire interval of orthogonality, or on a part of it : the problem of the conditions of this boundedness was first given by V.A.Strelkov [2]; its solution is extremely important, for many authors formulate their results on the basis of "if the orthogonal system is limited, then ....".

Our second task is to find the conditions under which a system of orthogonal polynomials is limited on the entire interval of orthogonality or on a part of it.

Another still more general problem, which is of direct importance in the study of infinite processes linked with orthogonal polynomials, is to find the order of the growth of these polynomials as a function of their number, without assuming their boundedness - this is the third task of this monograph.

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* The numbers in square brackets refer to the References, given at the end of the book.

POLYNOMIALS ORTHOGONAL ON CIRCLE

We examine polynomials orthogonal on the unit circle and polynomials orthogonal on a finite interval of the real axis.

In Chapters I and II we examine some properties of polynomials orthogonal on the unit circumference necessary for future working, and in doing so we use the results of our earlier monograph [1], but introduce new proofs for all the propositions, so that this monograph can be read independently of the earlier one; we take advantage of the opportunity to simplify some of the proofs.

Chapter III deals with inequalities on the entire circle, and Chapter IV with local inequalities. Using the inequalities found we conclude, in Chapter V, the conditions under which asymptotic formulae on the entire circle or on some of its arc are justified.

Chapter VI deals with the general theory of series by orthogonal polynomials, and in this the series are direct generalized power series; it is possible to generalize some classical properties of power series to the case of series of orthogonal polynomials.

Chapter VII examines the conditions necessary for the convergence of the Fourier-Chebyshev expansions of the given function. In particular we introduce the theorem of divergence, which links the convergence of the expansions of the given function into a Fourier-Chebyshev series and a Maclaurin series.

Chapter VIII studies an orthogonal system in terms not of the conditions imposed on the measure function but of its parameters. We deal with this question because in the very recent past M.G.Krein obtained extremely interesting and from the point of view of their application extremely important, results which can be viewed as direct analogues of the results given in this Chapter.

We examine the theory of polynomials orthogonal on the unit circumference in such detail because with the help of a simple formula they can be linked with polynomials orthogonal on the finite interval of a real axis.

Using this link we easily obtain inequalities and asymptotic formulae for polynomials orthogonal on the interval [- 1, + 1] from the corresponding results for polynomials orthogonal on the unit circle. This question is dealt with in Chapter IX.

In view of the fact that in many cases we obtained several conditions close to one another several Tables are given at the end of the book to enable the main results to be better reviewed and compared. For the readers convenience we have, moreover, in the notes given at the end of the book, given full formulation of all the theorems used in the text.

The present monograph is an attempt to develop and to apply to the solution of important problems the theorems of orthogonal polynomials and the ideas of V.A.Steklov, Bernstein, V.I.Smirnov, A.N.Kolmogorov, N.I.Akhierer, and Krein, and a number of foreign scientists - primarily Szëgö, P. Erdös, P.Turan, and G.Freud. Just how far we were successful in solving the problem we set ourselves, we cannot judge; we will, in any case, be extremely receptive to criticisms and comment of any kind.

We consider it a pleasant duty to express our deep thanks to N.I.Akhiezer, who read the manuscript of this book with great care, and who made a number of valuable comments.

CONTENTS

CHAPTER I

SOME PROPERTIES OF POLYNOMIALS ORTHOGONAL ON THE UNIT CIRCLE

Let the polynomials $\{\varphi_n(z)\}$ be orthonormal with respect to the measure $d\sigma(\theta)$ on the unit circle $|z| = 1$, i.e. the following conditions are satisfied:

$$\frac{1}{2\pi}\int_0^{2\pi} \varphi_n(e^{i\theta})\overline{\varphi_m(e^{i\theta})}\,d\sigma(\theta) = \begin{cases} 0, & n \neq m, \\ 1, & n = m, \end{cases} \tag{1.1}$$

$$\varphi_n(z) = \alpha_n z^n + \ldots, \quad \alpha_n > 0, \qquad (n = 0, 1, 2, \ldots),$$

where $\sigma(\theta)$ is bounded and non-decreasing over the interval $[0, 2\pi]$. The orthogonality conditions are readily seen to define an orthogonal system apart from a constant factor, which must have unit modulus because of the normality conditions: and finally, the conditions $\alpha_n > 0$ enable the system to be defined uinquely. We shall assume that the function $\frac{1}{2\pi}\sigma(\theta)$ characterizes a certain mass distribution over the interval $[0, 2\pi]$, the mass appearing on the set $e \subset [0, 2\pi]$ being equal to $\frac{1}{2\pi}\int_e d\sigma(\theta)$. The **density** of the mass distribution will be regarded as the derivative of a function $p(\theta)$ $\sigma'(\theta) = p(\theta)$ existing almost everywhere in $[0, 2\pi]$; if the function $\sigma(\theta)$ is absolutely continuous in $[0, 2\pi]$, we shall call $p(\theta)$ the **weight**. If $\sigma(\theta)$ has a discontinuity at a point θ_0 the quantity $\mu =$ $= \frac{1}{2\pi}\{\sigma(\theta_0 + 0) - \sigma(\theta_0 - 0)\}$ will be called the **concentrated mass**. We shall assume that $\sigma(\theta)$ has an infinite set of growth points, since otherwise we cannot form an infinite set of polynomials $\{\varphi_n(z)\}$.

The general theory of such polynomials in the case of an absolutely continuous function $\sigma(\theta)$ has been developed by Szegö [1], [5] and Smirnov [1], [3]. The general case has been discussed by Akhiezer and Krein [1] and more recently by the present author [1], [3], who has pointed to a connection with the trigonometric problem of moments, the theory of pseudo-positive and bounded functions and so on.

1.1 We shall first consider some algebraic properties of the polynomials $\{\varphi_n(z)\}$, needed later on.

(1) The following relationships hold* :

$$\left.\begin{aligned} \alpha_n\varphi_{n+1}(z) &= \alpha_{n+1}z\varphi_n(z) + \varphi_{n+1}(0)\varphi_n^*(z), \quad &(1.2)\\ \alpha_n\varphi_{n+1}^*(z) &= \alpha_{n+1}\varphi_n^*(z) + \overline{\varphi_{n+1}(0)}z\varphi_n(z) \quad &(1.2')\\ &(n=0,\ 1,\ 2,\ \ldots), \end{aligned}\right\}$$

where

$$\varphi_n^*(z) = z^n\bar{\varphi}_n\left(\frac{1}{z}\right), \quad \varphi_n^*(0) = \alpha_n, \quad (n=0,\ 1,\ 2,\ \ldots). \quad (1.2'')$$

Proof. We put

$$\frac{\varphi_n(z) - a_n z^n}{z^{n-1}} = \sum_{k=0}^{n-1}\mu_k^{(n)}\bar{\varphi}_k\left(\frac{1}{z}\right);$$

on multiplying both sides by $\varphi_k(z)$, setting $z=e^{i\theta}$ and integrating, we find for $k=0,\ 1,\ 2,\ \ldots,\ n-1$

$$\frac{1}{2\pi}\int_0^{2\pi}\varphi_n(e^{i\theta})\frac{\varphi_k(e^{i\theta})}{e^{i(n-1)\theta}}d\sigma(\theta) - \frac{a_n}{2\pi}\int_0^{2\pi}e^{i\theta}\varphi_k(e^{i\theta})\,d\sigma(\theta) = \mu_k^{(n)};$$

on introducing the notation

$$\frac{1}{2\pi}\int_0^{2\pi}e^{i\theta}\varphi_k(e^{i\theta})\,d\sigma(\theta) = \lambda_k, \qquad (k=0,\ 1,\ 2,\ \ldots), \quad (1.3)$$

we get

$$\mu_k^{(n)} = -\alpha_n\lambda_k, \quad \frac{\varphi_n(z)}{\alpha_n} = z^n - z^{n-1}\sum_{k=0}^{n-1}\lambda_k\bar{\varphi}_k\left(\frac{1}{z}\right),$$

whence

$$\frac{\varphi_{n+1}(z)}{\alpha_{n+1}} - \frac{z\varphi_n(z)}{\alpha_n} = -\lambda_n\varphi_n^*(z).$$

We find the λ_n by setting $z=0$:

$$\frac{\varphi_{n+1}(0)}{\alpha_{n+1}} = -\lambda_n\alpha_n, \quad \lambda_n = -\frac{\varphi_{n+1}(0)}{\alpha_n\alpha_{n+1}}, \quad (1.4)$$

whence (1.2) follows: to obtain (1.2') we replace z by $\frac{1}{z}$ and pass to the conjugate quantities.

(2) The following relationship holds**:

$$\alpha_{n+1}^2 - \alpha_n^2 = |\varphi_{n+1}(0)|^2, \qquad (n=0,\ 1,\ 2,\ \ldots). \quad (1.5)$$

We prove this by multiplying both sides of (1.2) by z^{-n-1}, $z=e^{i\theta}$, and integrating:

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* Cf. Szegö [1] § 11.4; Geronimus [1], § 3.

** Cf. Geronimus [1], § 4.3.

$$\frac{a_n}{2\pi}\int_0^{2\pi} e^{-i(n+1)\theta}\varphi_{n+1}(e^{i\theta})\,d\sigma(\theta) = \frac{a_{n+1}}{2\pi}\int_0^{2\pi} e^{-in\theta}\varphi_n(e^{i\theta})\,d\sigma(\theta) +$$

$$+\frac{\varphi_{n+1}(0)}{2\pi}\int_0^{2\pi} e^{-i\theta}\overline{\varphi_n(e^{i\theta})}\,d\sigma(\theta),$$

Using (1.3) and (1.4) we then obtain

$$\frac{a_n}{a_{n+1}} = \frac{a_{n+1}}{a_n} + \varphi_{n+1}(0)\,\overline{\lambda}_n = \frac{a_{n+1}}{a_n} - \frac{|\varphi_{n+1}(0)|^2}{a_n a_{n+1}}.$$

(3) The formal Fourier-Chebyshev expansion

$$\psi_0(\theta) \sim -\sum_{k=0}^{\infty} \frac{\overline{\varphi_{k+1}(0)}}{a_k a_{k+1}}\,\varphi_k(e^{i\theta}). \tag{1.6}$$

corresponds to the function $\psi_0(\theta) = e^{-i\theta}$.

For we have from (1.3), (1.4):

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-i\theta}\overline{\varphi_k(e^{i\theta})}\,d\sigma(\theta) = \overline{\lambda}_k = -\frac{\overline{\varphi_{k+1}(0)}}{a_k a_{k+1}}, \quad (k = 0, 1, 2, \ldots),$$

the sum

$$\sum_{k=r}^{p} |\lambda_k|^2 = \sum_{k=r}^{p} \left|\frac{\varphi_{k+1}(0)}{a_k a_{k+1}}\right|^2 = \frac{1}{a_r^2} - \frac{1}{a_{p+1}^2} \tag{1.6'}$$

being easily found by using (1.5).

(4) The Christoffel-Darboux formula is valid*:

$$K_n(x, y) = \sum_{k=0}^{n} \varphi_k(x)\overline{\varphi_k(y)} = \frac{\varphi_n^*(x)\overline{\varphi_n^*(y)} - x\overline{y}\varphi_n(x)\overline{\varphi_n(y)}}{1 - x\overline{y}} =$$

$$= \frac{\varphi_{n+1}^*(x)\overline{\varphi_{n+1}^*(y)} - \varphi_{n+1}(x)\overline{\varphi_{n+1}(y)}}{1 - x\overline{y}}, \quad (n = 0, 1, 2, \ldots). \tag{1.7}$$

To prove this, we find the sum

$$\frac{\varphi_m^*(x)\overline{\varphi_m^*(y)} - x\overline{y}\varphi_m(x)\overline{\varphi_m(y)}}{1 - x\overline{y}} + \varphi_{m+1}(x)\overline{\varphi_{m+1}(y)}. \tag{1.7'}$$

We find from (1.2), (1.2'), and (1.5) that

$$\left.\begin{aligned} \alpha_{m+1} x\varphi_m(x) &= \alpha_m\varphi_{m+1}(x) - \varphi_{m+1}(0)\,\varphi_m^*(x), \\ \alpha_{m+1}\varphi_m^*(x) &= \alpha_m\varphi_{m+1}^*(x) - \overline{\varphi_{m+1}(0)}\,x\varphi_m(x); \end{aligned}\right\} \tag{1.8}$$

on eliminating $\varphi_m^*(x)$ and $x\varphi_m(x)$, with the aid of these expressions, we easily find by using (1.2) and (1.5) that (1.7') is equal to

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* Cf. Szegö, 1, § 11.4: Geronimus, 1, § 8.

$$\frac{\varphi^*_{m+1}(x)\overline{\varphi^*_{m+1}(y)} - x\bar{y}\varphi_{m+1}(x)\overline{\varphi_{m+1}(y)}}{1-x\bar{y}};$$

We pass to the second form of formula (1.7) by making use of (1.2), (1.2') and (1.5).

(5) The polynomial $\varphi^*_n(z)$ is given by the formula*

$$\alpha_n\varphi^*_n(z) = K_n(z, 0) = \sum_{k=0}^{n} \varphi_k(z)\overline{\varphi_k(0)}, \qquad (n = 0, 1, 2, \ldots). \tag{1.9}$$

We prove this by setting $x = z,\ y = 0$ in (1.7).

(6) The following inequality applies **:

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{G_n(e^{i\theta})}{G_n(z_0)}\right|^2 d\sigma(\theta) \geqslant \frac{1}{K_n(z_0, z_0)}, \tag{1.10}$$

where the notation $G_n(z)$ will always be used for an arbitrary polynomial of degree not exceeding n.

We prove this by putting $G_n(z) = \sum_{k=0}^{n} a_k\varphi_k(z)$ and applying the Cauchy-Bunyakovskii inequality

$$|G_n(z_0)|^2 = \left|\sum_{k=0}^{n} a_k\varphi_k(z_0)\right|^2 \leqslant \sum_{k=0}^{n}|a_k|^2\cdot\sum_{k=0}^{n}|\varphi_k(z_0)|^2 =$$

$$= \frac{1}{2\pi}\int_0^{2\pi}|G_n(e^{i\theta})|^2 d\sigma(\theta)\cdot K_n(z_0, z_0).$$

The sign of equality may easily be seen to apply for the polynomial

$$G_n(z) = \varepsilon\frac{K_n(z, z_0)}{K_n(z_0, z_0)}, \quad |\varepsilon| = 1. \tag{1.10'}$$

(7) The inequality

$$\frac{1}{\alpha_n^2} = \frac{1}{K_n(0, 0)} \leqslant \frac{1}{2\pi}\int_0^{2\pi}\left|1 + \sum_{k=1}^{n}\beta_k e^{ik\theta}\right|^2 d\sigma(\theta), \tag{1.11}$$

applies, $\frac{1}{\alpha_n}\varphi^*_n(z)$ being the extremal polynomial.

This is simply proved by setting $z_0 = 0$ in (1.10).

* Cf. Szegö, [1], § 11.3
** Cf. Szegö, [1], § 11.3

(8) All the roots of the polynomial $\varphi_n(z)$ lie in the domain $|z|<1$. For we have from (1.7) for $|z|<1$

$$\alpha_0^2 \leqslant K_n(z, z) = \sum_{k=0}^{n} |\varphi_k(z)|^2 = \frac{|\varphi_n^*(z)|^2 - |z\varphi_n(z)|^2}{1-|z|^2} \leqslant \frac{|\varphi_n^*(z)|^2}{1-|z|^2}, \tag{1.12}$$

and consequently $\varphi_n^*(z) \neq 0$ with $|z|<1$, since

$$|\varphi_n^*(z)|^2 \geqslant \alpha_0^2(1-|z|^2), \qquad |z|<1; \tag{1.12'}$$

the absence of zeros of $\varphi_n(z)$ on the circle $|z|=1$ will be proved below.

(9) If we introduce the second order polynomials

$$\left.\begin{aligned} \psi_n(z) &= \frac{1}{2\pi c_0}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}[\varphi_n(e^{i\theta}) - \varphi_n(z)]\, d\sigma(\theta), \\ c_0 &= \frac{1}{2\pi}\int_0^{2\pi} d\sigma(\theta), \qquad (n=0, 1, \ldots), \end{aligned}\right\} \tag{1.13}$$

the functions $\dfrac{\psi_n^*(z)}{\varphi_n^*(z)}$ have the same Maclaurin coefficients in $\{z^k\}_0^n$ as the functions

$$F(z) = \frac{1}{2\pi c_0}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\sigma(\theta), \qquad |z|<1 \tag{1.14}$$

For we have

$$\begin{aligned} c_0\psi_n^*(z) &= -\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}[z^n\overline{\varphi_n(e^{i\theta})} - \varphi_n^*(z)]\, d\sigma(\theta) = \\ &= c_0F(z)\varphi_n^*(z) - \frac{z^n}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\overline{\varphi_n(e^{i\theta})}\, d\sigma(\theta); \end{aligned} \tag{1.15}$$

hence

$$\begin{aligned} c_0[F(z)\varphi_n^*(z) - \psi_n^*(z)] &= \\ &= z^n \cdot \frac{1}{2\pi}\int_0^{2\pi}\left\{1 + 2\sum_{k=1}^{\infty} z^k e^{-ik\theta}\right\}\overline{\varphi_n(e^{i\theta})}\, d\sigma(\theta) = \\ &= z^n \cdot 2\sum_{k=1}^{\infty} z^k \cdot \frac{1}{2\pi}\int_0^{2\pi} e^{-ik\theta}\overline{\varphi_n(e^{i\theta})}\, d\sigma(\theta) = O(z^{n+1}), \quad |z|<1. \end{aligned} \tag{1.15'}$$

Since $\varphi_n^*(z)\neq 0$ in the domain $|z|<1$, we get from (1.15'):

$$F(z)-\frac{\psi_n^*(z)}{\varphi_n^*(z)}=O(z^{n+1}),\qquad |z|<1. \tag{1.16}$$

An expression that will be useful later on follows from (1.15) and (1.12') with $|z|\leqslant r<1$

$$\left|F(z)-\frac{\psi_n^*(z)}{\varphi_n^*(z)}\right|\leqslant\frac{r^n}{c_0|\varphi_n^*(z)|}\cdot\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{e^{i\theta}+z}{e^{i\theta}-z}\right||\varphi_n(e^{i\theta})|\,d\sigma(\theta)\leqslant$$

$$\leqslant\frac{r^n(1+r)}{\sqrt{c_0\alpha_0}\sqrt{1-r^2}(1-r)}\sqrt{\frac{1}{2\pi}\int_0^{2\pi}|\varphi_n(e^{i\theta})|^2\,d\sigma(\theta)}\leqslant\frac{\sqrt{2}\,r^n}{(1-r)^{\frac{3}{2}}}. \tag{1.16}$$

(10) The relationship *:

$$c_0\{\psi_n(z)\varphi_n^*(z)+\varphi_n(z)\psi_n^*(z)\}=2z^n,\ (n=0,1,2,\ldots) \tag{1.17}$$

holds; in particular:

$$c_0\Re\left\{\frac{\psi_n^*(e^{i\theta})}{\varphi_n^*(e^{i\theta})}\right\}=\frac{1}{|\varphi_n^*(e^{i\theta})|^2}\qquad (n=0,1,2,\ldots). \tag{1.18}$$

We prove this by using the relationships

$$\left.\begin{aligned} c_0[F(z)\varphi_n^*(z)-\psi_n^*(z)]&=z^n\cdot\frac{1}{2\pi}\int_0^{2\pi}\frac{\zeta+z}{\zeta-z}\overline{\varphi_n(\zeta)}\,d\sigma(\theta),\\ &\qquad\qquad \zeta=e^{i\theta},\\ c_0[F(z)\varphi_n(z)+\psi_n(z)]&=\frac{1}{2\pi}\int_0^{2\pi}\frac{\zeta+z}{\zeta-z}\varphi_n(\zeta)\,d\sigma(\theta),\end{aligned}\right\} \tag{1.19}$$

following from (1.13) and (1.15). We find from these that

$$c_0[\psi_n(z)\varphi_n^*(z)+\varphi_n(z)\psi_n^*(z)]=\frac{1}{2\pi}\int_0^{2\pi}\frac{\zeta+z}{\zeta-z}\Omega_n(\zeta,z)\,d\sigma(\theta),$$

where, on the basis of (1.2") and (1.7):

$$\Omega_n(\zeta;z)=\varphi_n(\zeta)\varphi_n^*(z)-z^n\varphi_n(\zeta)\varphi_n(z)=$$

$$=\frac{\zeta^n-z^n}{\zeta^n}\varphi_n(\zeta)\varphi_n^*(z)+z^n[\overline{\varphi_n^*(\zeta)}\varphi_n^*(z)-\overline{\varphi_n(\zeta)}\varphi_n(z)]=$$

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* It follows from (1.17) that $\varphi_n(z)\neq 0$ with $|z|=1$.

$$= \frac{\zeta^n - z^n}{\zeta^n} \varphi_n(\zeta)\varphi_n^*(z) + \frac{z^n(\zeta - z)}{\zeta} \sum_{k=0}^{n-1} \overline{\varphi_k(\zeta)}\varphi_k(z).$$

We have further:

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{(\zeta^n - z^n)(\zeta + z)}{\zeta^n(\zeta - z)} \varphi_n(\zeta)\, d\sigma(\theta) = \frac{z^n}{\alpha_n};$$

we now find, on the basis of (1.3), (1.4), (1.2'):

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{\zeta + z}{\zeta} \sum_{k=0}^{n-1} \overline{\varphi_k(\zeta)}\varphi_k(z)\, d\sigma(\theta) =$$

$$= 1 + z\sum_{k=0}^{n-1} \varphi_k(z)\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{\zeta}\overline{\varphi_k(\zeta)}\, d\sigma(\theta) = 1 - z\sum_{k=0}^{n-1} \varphi_k(z)\frac{\overline{\varphi_{k+1}(0)}}{\alpha_k\alpha_{k+1}} =$$

$$= 1 - \sum_{k=0}^{n-1}\left\{\frac{\varphi_{k+1}^*(z)}{\alpha_{k+1}} - \frac{\varphi_k^*(z)}{\alpha_k}\right\} = 2 - \frac{\varphi_n^*(z)}{\alpha_n}.$$

whence (1.17) follows.

(11) We have with $k = 0, 1, \ldots, n$

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{-ik\theta}\, d\theta}{|\varphi_n^*(e^{i\theta})|^2} = \frac{1}{2\pi}\int_0^{2\pi} e^{-ik\theta}\, d\sigma(\theta) = c_k. \tag{1.20}$$

By property (9):

$$\frac{c_0}{2}\cdot\frac{\psi_n^*(z)}{\varphi_n^*(z)} = \frac{c_0}{2} + c_1 z + \ldots + c_n z^n + \sum_{k=n+1}^{\infty} c_k^{(n)} z^k,$$

the series being convergent in the domain $|z| \leqslant 1$; hence we have, on putting $c_{-k} = \overline{c}_k$, the equation

$$c_0 \mathfrak{R}\left\{\frac{\psi_n^*(e^{i\theta})}{\varphi_n^*(e^{i\theta})}\right\} = \frac{1}{|\varphi_n^*(e^{i\theta})|^2} = \sum_{k=-n}^{n} c_k e^{ik\theta} + \sum_{|k|>n} c_k^{(n)} e^{ik\theta},$$

from which (1.20) follows.

Thus, if $\sigma(\theta)$ is replaced by $\int_0^{\theta} \frac{d\theta}{|\varphi_n^*(e^{i\theta})|^2}$, all the moments $\{c_k\}_0^n$, and therefore all the polynomials $\{\varphi_k(z)\}_0^n$ remain unchanged.

(12) Function (1.14) has a positive real part inside the circle $|z| < 1$ almost everywhere on the circle

$z = e^{i\theta}$, $0 \leqslant \theta \leqslant 2\pi$, there exist the radial boundary values

$$\lim_{r \to 1-0} F(re^{i\theta}) = F(e^{i\theta}), \quad \lim_{r \to 1-0} \Re F(re^{i\theta}) = \frac{1}{c_0} p(\theta); \qquad (1.21)$$

or more precisely, at all points θ_0, at which there exists the first order generalized symmetric derivative

$$\sigma_{(1)}(\theta_0) = \lim_{h \to 0} \frac{\sigma(\theta_0 + h) - \sigma(\theta_0 - h)}{2h},$$

we have

$$\lim_{r \to 1-0} \Re F(re^{i\theta_0}) = \frac{1}{c_0} \sigma_{(1)}(\theta_0). \qquad (1.21')$$

For the real part of the function $F(re^{i\theta})$ is in fact given by the Poisson-Stieltjes integral

$$c_0 \Re F(re^{i\theta_0}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r\cos(\theta_0 - \theta) + r^2} d\sigma(\theta), \quad r < 1 \quad (1.22)$$

with a non-decreasing function $\sigma(\theta)$ and is therefore positive with $r < 1$; consequently by Smirnov's theorem* the function $F(z)$ belongs to class H_δ, $\delta < 1$ and thus has radial boundary values almost everywhere on the circle $|z| = 1$ **

Statement (1.21') follows from the familiar results regarding the summation of Fourier-Stieltjes series by the Abel-Poisson method.***

1.2. We now consider some limiting relationships, valid as $n \to \infty$.

We introduce the space $L_r^\sigma(0, 2\pi)$ of complex-valued periodic functions $f(\theta)$ with the usual definition of norm

$$\|f\|_r^\sigma = \sqrt[r]{\frac{1}{2\pi} \int_0^{2\pi} |f(\theta)|^r d\sigma(\theta)} < \infty \qquad (r > 0), \quad (1.23)$$

if $d\sigma(\theta) = d\theta$, the superscript "$\sigma$" will be omitted for the space.

13. The finite or infinite limit

$$\lim_{n \to \infty} \alpha_n = \alpha \leqslant \infty; \qquad (1.24)$$

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* Cf. Smirnov [3]; Cf. also Privalov [1] Ch.II, 4.5.

** Cf. Privalov [1], Ch. II, 1.1, 2.1

*** Cf. Zygmund [1], §§ 2.14, 3.44

exists. It is sufficient to observe here that in accordance with (1.5) $0 < \alpha_n \leqslant \alpha_{n+1}$.

(14) We have for the function $\psi_0(\theta) = e^{-i\theta}$

$$\sum_{k=0}^{\infty} |\lambda_k|^2 = \sum_{k=0}^{\infty} \left| \frac{\varphi_{k+1}(0)}{\alpha_k \alpha_{k+1}} \right|^2 = \frac{1}{\alpha_0^2} - \frac{1}{\alpha^2} = \tag{1.25}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} |\psi_0(\theta)|^2 \, d\sigma(\theta) - \frac{1}{\alpha^2},$$

where

$$\lim_{n\to\infty} \min_{G_n} \| \psi_0(\theta) - G_n(e^{i\theta}) \|_2^\sigma = \frac{1}{\alpha} \geqslant 0. \tag{1.25'}$$

(15) The following statements are equivalent:

(a) the function $\sigma'(\theta)$ is summable, i.e.

$$\int_0^{2\pi} \ln \sigma'(\theta) \, d\sigma > -\infty; \tag{1.26}$$

(b) the system of polynomials $U = \{\varphi_n(e^{i\theta})\}_0^\infty$, and so also the system of powers $\{e^{in\theta}\}_0^\infty$ is non-closed in L_2^σ;

(c) the function $\psi_0(\theta) = e^{-i\theta}$ cannot be approximated by polynomials in the metric L_2^σ with any degree of accuracy

(d) the finite limit

$$\lim_{n\to\infty} \alpha_n = \alpha; \tag{1.27}$$

exists:

(e) the series $\sum_{k=0}^{\infty} |\varphi_k(z)|^2$ is convergent in the domain $|z| < 1$ except at a single point:

(f) there exists the subsequence $\{\varphi^*_{n_\nu}(z)\}$, bounded in $|z| < 1$ *) except at a single point.*

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* The equivalence of conditions (a) and (b) was shown by Kolmogorov [1]. [2] and by Krein [1]. The condition of non-closedness of a system of orthogonal polynomials in the space L_r^σ with $r \geqslant 1$ was found by Akhiezer [1] in the case of orthogonality on the unit circle; in the general case of orthogonality on a Jordan rectifiable contour the condition was discovered by the author [2] for $r \geqslant 1$ and by Tumarkin [1] for $r > 0$.

Suppose that the system U is not closed, and therefore not complete, in $\overset{\circ}{L}_2$; then there exists a function $\varphi(\theta) \in L_2^{\sigma}$, not equivalent to zero and orthogonal to all the functions of system U, i.e. we can write

$$\frac{1}{2\pi}\int_0^{2\pi} \overline{\varphi(\theta)}\, e^{ik\theta}\, d\sigma(\theta) = 0, \quad (k = 0, 1, 2, \ldots). \tag{1.28}$$

On multiplying each of these integrals by z^{-k-1}, $|z| > 1$ and summing over k, we get

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{\overline{\varphi(\theta)}\, d\sigma(\theta)}{e^{i\theta} - z} = 0, \qquad |z| > 1. \tag{1.28'}$$

We introduce the notation

$$\lambda(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}\, d\tau(\theta)}{e^{i\theta} - z}, \qquad d\tau(\theta) = e^{-i\theta}\overline{\varphi(\theta)}\, d\sigma(\theta); \tag{1.29}$$

since

$$\frac{1}{2\pi}\int_0^{2\pi} |d\tau(\theta)| = \frac{1}{2\pi}\int_0^{2\pi} |\varphi(\theta)|\, d\sigma(\theta) \leqslant \|\varphi\|_2^{\sigma} \cdot \sqrt{c_0} < \infty,$$

$\tau(\theta)$ is a function of bounded variation.

The function $\lambda(z)$ defined by (1.29) is analytic and regular in the domain $|z| < 1$; since it is given by an integral of the Cauchy-Stieltjes type (1.29) and $\lambda(z) \equiv 0$ in the domain $z > 1$, expression (1.29) is in fact a Cauchy-Stieltjes integral; thus $\lambda(z) \in H_1$, $|z| < 1$ and almost everywhere in $[0, 2\pi]$ the boundary values exist:

$$\lambda(e^{i\theta}) = \lim_{r \to 1-0} \lambda(re^{i\theta}) = \tau'(\theta) = e^{-i\theta}\overline{\varphi(\theta)}\,\sigma'(\theta), \tag{1.30}$$

with the boundary function $\lambda(e^{i\theta})$ satisfying *

$$\int_0^{2\pi} \ln|\lambda(e^{i\theta})|\, d\theta > -\infty, \qquad \int_0^{2\pi} |\lambda(e^{i\theta})|\, d\theta < \infty. \tag{1.31}$$

We find by using (1.30) that $\ln\{|\varphi(\theta)|\,\sigma'(\theta)\} \in L_1$, whence **

$$-\infty < \int_0^{2\pi} \ln^+ \sigma'(\theta)\, d\theta + \int_0^{2\pi} \ln^+ \{|\varphi(\theta)|^2 \sigma'(\theta)\}\, d\theta +$$

$$+ \int_0^{2\pi} \ln^- \sigma'(\theta)\, d\theta + \int_0^{2\pi} \ln^- \{|\varphi(\theta)|^2 \sigma'(\theta)\}\, d\theta < +\infty.$$

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* Cf. Privalov [1], Ch. II. § 5.

** We have put $\ln^+ a = \begin{cases} \ln a, & a \geqslant 1, \\ 0, & 0 < a < 1, \end{cases}$ $\ln^- a = \ln a - \ln^+ a.$

On the other hand, we have from the conditions $\sigma'(\theta) \in L_1$, $\varphi(\theta) \in L_2^\sigma$

$$\left.\begin{aligned} &\int_0^{2\pi} \ln^+ \sigma'(\theta)\, d\theta < \int_0^{2\pi} \sigma'(\theta)\, d\theta < +\infty, \\ &\int_0^{2\pi} \ln^+ \{|\varphi(\theta)|^2 \sigma'(\theta)\}\, d\theta < \int_0^{2\pi} |\varphi(\theta)|^2 \sigma'(\theta)\, d\theta < \\ &\qquad\qquad < \int_0^{2\pi} |\varphi(\theta)|^2\, d\sigma(\theta) < +\infty, \end{aligned}\right\} \tag{1.32}$$

and consequently

$$-\infty < \int_0^{2\pi} \ln^- \sigma'(\theta)\, d\theta + \int_0^{2\pi} \ln^- \{|\varphi(\theta)|^2 \sigma'(\theta)\}\, d\theta.$$

Since both terms are negative, we arrive at the inequalities

$$-\infty < \int_0^{2\pi} \ln^- \sigma'(\theta)\, d\theta, \qquad -\infty < \int_0^{2\pi} \ln \sigma'(\theta)\, d\theta < +\infty.$$

Now let (1.26) be given: we have from (1.11):

$$\frac{1}{\alpha_n^2} = \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\varphi_n^*(e^{i\theta})}{\alpha_n} \right|^2 d\sigma(\theta) \geqslant \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\varphi_n^*(e^{i\theta})}{\alpha_n} \right|^2 \cdot \sigma'(\theta)\, d\theta;$$

on using the fact that the geometric mean of a function does not exceed its arithmetic mean, we find that

$$\frac{1}{\alpha_n^2} \geqslant \exp\left\{ \frac{2}{2\pi} \int_0^{2\pi} \ln \left| \frac{\varphi_n^*(e^{i\theta})}{\alpha_n} \right| d\theta \right\} \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} \ln \sigma'(\theta)\, d\theta \right\} = \tag{1.33}$$

$$= \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} \ln \sigma'(\theta)\, d\theta \right\},$$

since $\varphi_n^*(0) = \alpha_n$; hence, by (1.26),

$$\frac{1}{\alpha^2} = \lim_{n\to\infty} \frac{1}{\alpha_n^2} \geqslant \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} \ln \sigma'(\theta)\, d\theta \right\} > 0, \tag{1.34}$$

whence (d) follows.

The equivalence of (c) and (d) follows from (1.25'); and (b) follows from (c).

It follows from (1.12') that the sequence $\{\varphi_n^*(z)\}$ is uniformly bounded from below in the domain $|z| \leqslant r < 1$; by Montel's theorem * there exists a subsequence of polynomials $\{\varphi_{n_\nu}^*(z)\}$, for which we have uniformly

* See Montel [1], § 17.