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Robert C. McOwen 著

偏微分方程

—— 方法及应用

Partial Differential Equations

Methods and Applications

Pearson Education

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Preface

This book has evolved from a two-term graduate course in partial differential equations which I have taught at Northeastern University many times since 1980. The first term is intended to give the student a basic and classical introduction to the subject, including first-order equations by the method of characteristics and the linear second-order equations which arise in mathematical physics: the wave equation, Laplace equation, and heat equation. All of this material is more than adequately covered by many textbooks which are readily available. The second term, however, is intended to introduce the student to a wide variety of more modern methods, especially the use of functional analysis, which has characterized much of the recent development of partial differential equations. This latter material is not as readily available, except in a number of specialized reference books. This textbook is intended to bridge this gap by providing the student with a basic introduction to the subject and an exposure to some of the more modern methods.

As with any other book on such a broad and diverse subject as partial differential equations, I have had to make some difficult decisions concerning content and exposition. I make no apologies for these decisions, but I do acknowledge that other choices might have been made. For example, this text begins with the method of characteristics and first-order equations; although other texts often omit or slight this material in preference to the treatment of second-order equations, I have chosen to include it, and even emphasize its constructive aspects, because I feel it offers motivation and insights which are valuable in the study of higher-order equations. Indeed, the method of characteristics leads naturally to the Cauchy problem for higher-order equations, as well as the classification of second-order equations, which I treat in Chapter 2 (along with a discussion of generalized solutions). Following this momentum, I decided to treat the wave equation before Laplace's equation, even though this causes the use of eigenfunctions in a bounded domain to be delayed until the next chapter. Similarly, I have chosen to treat the heat equation after the Laplace equation for reasons of the maximum principle; of course, a bonus is that eigenfunction expansions are available for the heat equation in a bounded domain. Other texts treat these three second-order equations in different orders, and they all have their own reasons for doing so.

Exposure to the use of functional analysis begins in Chapter 6 with a rapid survey of the basic definitions and tools needed to study linear operators on Banach and Hilbert spaces. The Sobolev spaces are introduced as early as possible, as are their application to obtain weak solutions of various Dirichlet problems. This early application of Sobolev spaces establishes the weak solution as a theme which recurs through much of remainder of the book; it also emphasizes the usefulness of the functional analytic approach before encountering the more subtle issues of weak convergence, continuous imbeddings, and compactness.

The theme of weak solutions is picked up again in Chapter 7, in the context of differential calculus on Banach spaces. The variational method of finding a weak solution by optimizing a functional, possibly with constraints, is applied to several problems, including the eigenvalues of the Laplacian. The forum of differential calculus also enables us to introduce, at this point, the contraction mapping principle, the inverse and implicit function theorems, a discussion of when they apply to Sobolev spaces, and an application to the prescribed mean curvature equation.

The issue of the regularity of weak solutions is taken up in Chapter 8, where the basic elliptic L^2-estimates are obtained by Fourier analysis on a torus, and transplantation to open domains. It is also natural, at this point, to discuss maximum principles for elliptic operators, and then the issues of uniqueness and solvability for linear elliptic equations.

Chapter 9 consists of "two additional methods." The first of these, the Schauder fixed point theory, is presented and then illustrated with its application to the stationary Navier-Stokes equations; this application returns us to our theme of weak solutions in Sobolev spaces, and also builds on the discussion of the Stokes system in Chapter 6. The second "additional method" is the use of semigroups of operators on a Banach space to describe the dynamics of evolutionary partial differential equations. We first discuss systems of ordinary differential equations as a finite-dimensional example; this helps to motivate the ensuing discussion for partial differential equations, which is well-seasoned with examples. This treatment of semigroups is very brief, but serves the purpose of setting the stage for the hyperbolic and parabolic equations and systems which are studied in Chapters 10, 11, and 12.

Although Chapters 6-9 emphasize the development of tools and methods, I have tried to provide sufficient applications to motivate and illustrate the theory as it unfolds. However, beginning in Chapter 10, the focus switches from methods to applications, and developing the theories of hyperbolic systems conservation laws in one space dimension (Chapter 10), linear and nonlinear diffusion (Chapter 11), linear and nonlinear waves (Chapter 12), and nonlinear elliptic equations (Chapter 13) as far as possible in this limited space. I have, of course, needed to severely "limit the budget" in each of these last four chapters, but I hope I have given the flavor and some background on each topic, enough to enable the interested student to consult more detailed and comprehensive treatments.

Although I have made certain choices for the order, I have tried to make the exposition flexible enough to allow for the individual instructor to make changes without too much difficulty. For example, to enable the introduction of the spherical mean in connection with the Laplace equation instead of the wave equation, I have made Section 3.2a self-contained. This means that it is possible to re-order the material following Chapter 2: the one-dimensional wave equation, then Laplace's equation (with Section 3.2a added to Section 4.1d), and then the n-dimensional wave equation. Similarly, although I felt the need to collect together all of the linear functional analysis and Sobolev space theory in Chapter 6, it is possible to only discuss the results for $H_0^{1,2}(\Omega)$ in order to more quickly study the Dirichlet problems in Chapters 7, 8, and 9. Another example would be to jump into Chapter 10 after only a minimal amount of Banach space theory and the contraction mapping principle.

I have tried to include a large number of exercises. Some of these exercises are fairly routine applications of the material covered in the text. Other exercises are designed to supply some steps which are omitted from the exposition in the text; this not only helps to streamline the exposition, but it also engages the student more actively in the learning experience. Still other exercises are intended to give the student a brief exposure to related topics which have been reluctantly omitted from the textual exposition, casualties of more hard choices of mine. When I teach this course, I usually assign many exercises, including some of each type. On the other hand, the instructor may choose to use lecture time to "solve" all omitted steps of proofs, and/or pursue some of the omitted topics. In any case, hints and solutions of selected exercises are provided after Chapter 13; I hope the instructor and student find these useful.

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Contents

Introduction

A *partial differential equation* (abbreviated *PDE*) is an equation involving a function u of several variables, and its partial derivatives. For example,

$$u_t = k u_{xx}, \tag{1}$$

$$u_{tt} = c^2 u_{xx}, \tag{2}$$

$$u_{xx} + u_{yy} = 0, \tag{3}$$

are all partial differential equations for functions of two variables which are familiar from undergraduate courses on differential equations: (1) is the one-dimensional ***heat equation*** in which u represents the temperature of a heat-conducting rod having k as its heat conductivity; (2) is the one-dimensional ***wave equation*** in which u represents the displacement of a vibrating string from its equilibrium position and c represents the speed of wave propagation; (3) is the two-dimensional ***Laplace equation*** which arises as a steady-state condition in heat conduction problems and occurs in many other problems of analysis and mathematical physics.

More generally, a PDE for a function $u(x_1, \ldots, x_n)$ is of the form

$$F(x_1, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n}, u_{x_1 x_1}, u_{x_1 x_2}, \ldots) = 0. \tag{4}$$

The *order* of (4) is the order of the highest derivative occuring in the equation. Moreover, the equation is ***linear*** if it depends linearly on u and its derivatives; if all derivatives of u occur linearly with coefficients depending only on x, then the equation is ***semilinear***; and if all highest-order derivatives of u occur linearly with coefficients depending only on x, u, and lower-order derivatives of u, then the equation is ***quasilinear***. The equations (1)-(3) are all second-order linear equations. A simple example of a first-order PDE is

$$u_t + a(u) u_x = 0. \tag{5}$$

When $a(u) \equiv a$ is a constant, (5) is a linear equation called the ***transport equation***. In general, (5) is a quasilinear equation; for example, when $a(u) \equiv u$, the equation is called the ***inviscid Burgers' equation*** which arises in the study of a one-dimensional stream of particles or fluid having zero viscosity. An example of a first-order nonlinear PDE is

$$u_x^2 + u_y^2 = c^2, \tag{6}$$

which is the ***eikonal equation of geometric optics.***

We can generalize the equations (1)-(3) to higher dimensions if we introduce the *Laplacian* or *Laplace operator* $\Delta = \partial^2/\partial x_1^2 + \ldots + \partial^2/\partial x_n^2$. Then we can write

$$u_t = k\Delta u, \tag{7}$$

$$u_{tt} = c^2\Delta u, \tag{8}$$

$$\Delta u = 0. \tag{9}$$

Equation (7) represents the diffusion of heat through an n-dimensional body; equation (8) represents surface waves if $n = 2$, and sound or light waves if $n = 3$; and equation (9) is the n-dimensional Laplace equation. Equations (7) and (8) are both ***evolution equations*** because they describe phenomena which may change with time; equation (9) on the other hand is satisfied by ***steady-state*** (time independent) solutions of (7) and (8).

In order for a PDE to have a ***unique*** solution, we must impose additional conditions, sometimes called ***side conditions,*** on the solution; these are usually in the form of ***initial conditions*** or ***boundary conditions*** or some combination of the two. This is certainly familiar from ordinary differential equations (abbreviated *ODEs)* where a first-order equation requires an initial condition, and a second-order equation requires either two initial conditions, or a boundary condition at each end of a finite interval. The need for side conditions is also evident from the physical models. For example, we cannot know the temperature of a cooling body if we do not know its initial temperature; but even knowing the initial temperature will not be enough unless we also monitor what happens to the temperature on the boundary of the body. In this case, we must impose upon (7) an initial condition and a boundary condition: the result is called an ***initial/boundary value problem.*** For other PDEs we may be able to consider a pure ***initial value problem*** or pure ***boundary value problem.***

The values assigned to the side conditions are called the ***data.*** A PDE with side conditions is ***well posed*** if it admits a unique solution for any values assigned to the data. (Actually, well-posedness should also require the solution to depend continuously on the data, as we shall discuss later.)

Of course, the equations (7)-(9) are very special, and we may wonder how to handle more complicated equations which arise in applications. To begin, we can add a function f to these equations to obtain ***nonhomogeneous*** equations. For example, the nonhomogeneous Laplace equation (sometimes called the *Poisson equation*)

$$\Delta u = f \tag{10}$$

arises in various field theories such as electrostatics. Similarly, considerations of heat sources and external forcing terms in (7) and (8) respectively lead to the nonhomogeneous heat and wave equations

$$u_t - k\Delta u = f, \tag{11}$$

$$u_{tt} - c^2\Delta u = f. \tag{12}$$

These equations may be modified further if additional considerations are in effect; for example, consideration of a restoring force in (8) leads to the *Klein-Gordon equation*

$$u_{tt} - c^2\Delta u + m^2 u = 0, \tag{13}$$

which arises in quantum field theory with m denoting mass, and consideration of a damping or *dissipation term* in (13) leads to

$$u_{tt} - c^2\Delta u + \alpha u_t + m^2 u = 0, \tag{14}$$

which in one space dimension is called the *telegrapher's equation* because it governs electrical transmission in a telegraph cable when current may leak to the ground.

Another important second-order linear equation which arises in quantum mechanics is *Schrödinger's equation*

$$u_t = i(\Delta u + V(x)u) \tag{15}$$

in which $i = \sqrt{-1}$ indicates that the solution $u(x,t)$ must be complex-valued; the function $V(x)$ is called the *potential.*

So far, the second-order equations we have mentionned are all linear. This is not surprising, since the theory is simpler and for certain modeling purposes, a linear equation may suffice. But in other situations the nonlinear character of the problem is important and even essential. For example, if we allow f in (10)-(12) to depend on u, then we obtain the *semilinear Poisson equation*

$$\Delta u = f(x,u), \tag{16}$$

the *semilinear heat equation*

$$u_t - k\Delta u = f(x,t,u), \tag{17}$$

and the *semilinear wave equation*

$$u_{tt} - c^2 \Delta u = f(x, t, u). \tag{18}$$

A specific instance of (16) is the *conformal scalar curvature equation*

$$\Delta u + K(x) e^{2u} = 0 \qquad (n = 2), \tag{19a}$$

$$\Delta u + K(x) u^{\frac{n+2}{n-2}} = 0 \qquad (n \geq 3), \tag{19b}$$

which occurs in differential geometry when studying the scalar curvature of Riemannian metrics which are conformally Euclidean: for $n = 2$ the metric $e^{2u}(dx^2 + dy^2)$ will have Gauss curvature $K(x, y)$ if u satisfies (19a). Specific instances of (18) are the *semilinear Klein-Gordon equation*

$$u_{tt} - c^2 \Delta u + m^2 u + \gamma u^p = 0 \qquad (p \text{ an integer } \geq 2), \tag{20}$$

which arises in quantum field theory with γ denoting a "coupling constant," and the *sine-Gordon equation*

$$u_{tt} - c^2 \Delta u + \sin u = 0, \tag{21}$$

which also arises in quantum field theory, but was first studied in differential geometry in connection with surfaces of constant curvature. If we allow dissipation in (20) or (21), we get the *dissipative Klein-Gordon* or *dissipative sine-Gordon equations*

$$u_{tt} - c^2 \Delta u + \alpha u_t + m^2 u + \alpha u^p = 0, \tag{22}$$

$$u_{tt} - c^2 \Delta u + \alpha u_t + \sin u = 0. \tag{23}$$

A semilinear version of (15) is the *cubic Schrödinger equation*

$$u_t = i(\Delta u + \sigma |u|^2 u) \qquad \sigma = \pm 1, \tag{24}$$

which arises in nonlinear optics, and also the study of deep water waves.

Equations also arise in applications which are not semilinear. For example, in differential geometry the *minimal surface equation*

$$\operatorname{div}\left(\frac{\nabla u}{(1 + |\nabla u|^2)^{1/2}}\right) = 0 \tag{25}$$

is a second-order quasilinear equation for a graph $z = u(x, y)$ which has the smallest surface area for a given boundary curve; for example, soap films are minimal surfaces. In (25), "div" denotes the *divergence* of the vector field $(1+|\nabla u|^2)^{-1/2}\nabla u$, and (25) is said to be in *divergence form.* A quasilinear PDE arising in physics is the *porous medium equation*

$$u_t = k \operatorname{div}(u^\gamma \nabla u), \tag{26}$$

where $k > 0$ and $\gamma > 1$ are constants; this equation governs the seepage of a fluid through a porous medium (e.g., water through soil). An example of a nonlinear PDE which is not quasilinear is the *Monge-Ampère equation*

$$\det(u_{ij}) = f(x,u), \tag{27}$$

which arises in differential geometry; here the second-order derivatives $u_{ij} = \partial^2 u/\partial x_i \partial x_j$ occur in a nonlinear way.

Equations arising in applications need not be restricted to second-order. The *biharmonic equation*

$$\Delta^2 u \equiv \Delta(\Delta u) = 0 \tag{28}$$

is a fourth-order linear equation which occurs in elasticity theory, whereas the *Korteweg de Vries equation* (or *KdV equation*)

$$u_t + cuu_x + u_{xxx} = 0 \tag{29}$$

is a third-order quasilinear equation which was first encountered in the study of shallow water waves.

If u is replaced by a vector-valued function $\vec{u}(x_1,\ldots,x_n)$ and F is also vector-valued, then (4) becomes a *system* of differential equations. For example, if $A(x,t)$ and $B(x,t)$ are $N \times N$ matrix-valued functions, and $\vec{c}(x,t)$ is a vector-valued function, then

$$A(x,t)\vec{u}_x + B(x,t)\vec{u}_t = \vec{c}(x,t) \tag{30}$$

is a linear first-order system. An example of (30) is *Maxwell's equations* from electromagnetism theory in $\mathbf{R}^3$:

$$\vec{E}_t - \operatorname{curl}\vec{H} = 0 \qquad \vec{H}_t + \operatorname{curl}\vec{E} = 0, \tag{31}$$

where $\vec{E}$ denotes the electric field and $\vec{H}$ the magnetic field. Notice that (31) is a system of six equations in the six unknowns $(\vec{E},\vec{H})$.

A natural example of a nonlinear first-order system occurs in fluid mechanics when the balancing of forces according to Newton's law produces *Euler's equations*

$$\vec{u}_t + (\vec{u}\cdot\nabla)\vec{u} + \frac{1}{\rho}\nabla p = 0 \tag{32}$$

for an inviscid (no viscosity) fluid with velocity field $\vec{u}$, pressure p, and density ρ. In dimensions $n = 1$, 2, or 3, $\vec{u}$ has n components, and is a function of the $n+1$ variables (x,t); moreover $(\vec{u}\cdot\nabla)\vec{u}$ denotes the n vector whose jth component is $\sum_i u_i \partial u_j/\partial x_i$. Notice that (32) is a system of n equations in the $n+2$ unknowns $(\vec{u}, p, \rho)$, and so is underdetermined. One additional equation that must be coupled with (32) expresses the compressibility properties of the fluid. If the fluid is *incompressible*, this equation is div $\vec{u} = 0$, whereas for *compressible* fluids we must use $\rho_t +$ div $(\rho\vec{u}) = 0$ which expresses *conservation of mass*. Of course, these equations coincide for *homogeneous* fluids, i.e., constant density ρ, and coupling with (32) yields a well-posed system. More general than constant density is *isentropy*, which means that pressure is a known function of the density: $p = p(\rho)$, which is called an *equation of state*. Coupling (32) with an equation of state and either incompressibility or conservation of mass yields a well-posed system. For $n \geq 2$, the Euler equations are used to model vortices and turbulence. Natural questions to ask are whether smooth initial data produces a smooth solution at least for a short period of time, and whether an initially smooth solution can develop singularities.

If one considers a viscous fluid, then we must replace (32) by the *Navier-Stokes equations*

$$\vec{u}_t + (\vec{u}\cdot\nabla)\vec{u} + \frac{1}{\rho}\nabla p = \nu\Delta\vec{u}, \tag{33}$$

where ν is the viscosity and Δ operates on each component of $\vec{u}$. Although (33) may seem more complicated than (32), the viscosity term actually has a tempering influence on the solutions which is not present in (32). A natural question to ask is whether the viscosity enables a smooth solution to exist for all time. This is known to be true for $n = 2$, but as yet has not been proved for $n = 3$, in spite of ample physical intuition! Another important question in fluid dynamics is how well does (33) approximate (32) as $\nu \to 0$? This may be useful in modeling turbulence.

This comparison of (32) and (33) touches upon some general themes which we shall encounter for evolution equations. Does a solution exist at least for a short time? If so, does it exist for all time, and can we describe its behavior asymptotically as $t \to \infty$? If the solution fails to exist globally in time, is this due to a "blow-up" (the values of the solution approaching infinity), or a "gradient catastrophe" (the values of the spatial derivatives becoming infinite), or some more complicated singularity at a finite time?

On the other hand, for time-independent or "stationary" PDEs such as the Laplace equation, the conformal scalar curvature equation, and the minimal surface equation, we may ask: What boundary conditions are appropriate? Does a solution exist? Is it unique? How smooth is the solution?

In this book, we shall develop some of the methods used to study such linear and nonlinear PDEs, and apply these methods to obtain conclusions about their solvability, and the behavior of the solutions.