Mathematical Theories of Traffic Flow

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Preface

The population of the earth has been persistently increasing throughout recorded history, but only within the twentieth century has its size become an important obstacle to orderly civilization. One of the problems created by this growth, which has proved to be of some mathematical interest, is that of congestion. On land and in the air, in vehicles and on foot, people now get in each others' way to an extent far surpassing that of any previous age. There may have been overcrowding in ancient Rome or Elizabethan London, but it can hardly have constituted the hazard, the inconvenience, or the expense which it does today. We see congestion not only in transportation, but in virtually every aspect of modern life: communication, urban development, commercial organization, reticulation of utilities, mass production, and perhaps even agriculture.

The scientific study of congestion, whether intended to describe or to ameliorate, has been a natural consequence of man's enforced interest in his increasingly overcrowded world. The most fully developed mathematical theory of congestion is *queueing theory*, which deals with accumulation at a fixed point, caused by the need for *service*, and changing with the passage of *time*. The subject is more than fifty years old and is now being extended very vigorously, both in depth of formulation and in breadth of application.

As a source of congestion, the motor vehicle occupies a unique position, both from the practical and from the mathematical point of view. Estimates of the importance of transportation by car are difficult to make, but we can be sure that in an industrialized society the effect is enormous, whether measured economically, politically, in terms of public health, psychologically,* industrially, or purely as a fraction of transportation in general. Some of the central concepts of the present book, such as traffic delay, traffic flow, and traffic density, are popular concepts, and quite justifiably so. Few areas of applied mathematics have such widespread and directly intuitive importance in our lives.

On the mathematical side we find many genuinely interesting aspects of traffic flow. The development in the past decade of a substantial theory of vehicular movement has come not only from the need to understand more exactly the empirical results of the traffic engineering profession, but also as a natural extension of the theory of queues. We consider simultaneously the dimensions of *time* and *space*, and permit delay to arise partially from waiting (sequentially in a single dimension) but more generally from mere proximity in the time-space plane. Many intrinsically attractive mathematical questions arise from the traffic-theoretic formulation, some of which are not yet answered. Although the problems are difficult to formulate and still more difficult to solve, there is by now a considerable literature in traffic flow theory and the subject is ready for systematic exposition.

Perhaps it is unnecessary to add that we are not trying to "solve the traffic problem," any more than a hydrodynamicist is trying to solve the water problem or an entomologist the bug problem. The Traffic Problem, like The Toll of the Road, are expressions quite innocent of precise significance. Our mathematical theories should define, characterize, and describe one specific phenomenon: vehicular traffic. We must assume that these investigations, like those of classical applied mathematics, will lead naturally to improved techniques for practical understanding and control of the subject being studied.

As a simple consequence of its immaturity, traffic flow theory has been developed by research workers of widely varying interests:

* A simple anomolous by-product, the traffic accident, engages public attention to an astonishing degree, obscuring industrial and domestic accidents, as well as other more significant contributions to mortality.

mathematicians, physicists, traffic engineers, economists and, more recently, practitioners of operations research. In this circumstance the reader will not be surprised to learn that the field is sprawling, diffuse, and in many ways rather baffling. There is no general agreement on notation or terminology, most of which has been inherited from the traffic engineer. There is very little agreement on methodology, or on which quantities are significant, or on how these quantities should be measured. Armed with the Poisson distribution and a sufficient interest, nearly anyone can find some original, and possibly valuable, theme in traffic flow.

In this, the first attempt to justify the theory as a sensible part of applied mathematics, I have selected fairly ruthlessly from the literature in order to bring out the fundamental relationship between traffic flow theory and the classical subjects of queueing theory, stochastic processes, and mathematical probability. The present volume reflects my belief that the greatest development of traffic theory will take place in these directly connected areas, rather than by analogical variation of equations which apply to other substances.*

It is important to find meaningful middle ground between pure mathematics and traffic engineering. It should be possible to show traffic engineers the usefulness of theoretical analysis, and at the same time the mathematician should find systems worthy of his consideration. Keeping both of these requirements in mind, I use the framework of mathematical demonstration, while dealing with concepts of direct traffic-theoretic significance. A plausible although possibly heuristic, proof is preferred both to empirical dogmatism and to decorative abstraction.

Although I employ the terminology of roads and vehicles, the attentive reader will find many portions of the book with wider application. The postulates of Chapter 3, intended to characterize road traffic, are often partly laid aside, so that the structure is

* There are some references to alternatives in the supplementary list which follows Chapter 7. Of these, the "Boltzmann-like" system is the most substantial.

in fact more general. The precise degree of generality varies from section to section, and in a few cases the identification with vehicles is indeed tenuous.

The reader is assumed to have some familiarity with mathematical probability. Chapter 1 is designed as a convenient collection of results for reference, but is self-contained and may be used as a compact introduction to the subject. Chapter 2 stands in the same relation to the theory of queues. No other knowledge is required except for undergraduate mathematics and perhaps an intuitive idea of typical vehicular behavior in industrialized nations.

I owe a great deal to Robert M. Oliver for discussions on traffic flow theory. Chapters 3 and 4 in particular have benefited from our many interesting arguments, and it is a pleasure for me to acknowledge this debt to an esteemed colleague. The manuscript was read by Gordon Newell and Alan Miller, and their comments have been very helpful in preparing the final version. I have also received useful advice and comments from E. Farnsworth Bisbee, Leo Breiman, Leslie Edie, William Jewell, George Weiss, John R. B. Whittlesey, and from the participants in a series of seminars organized by Serge Goldberg of the Ministère des Travaux Publics and given at the Ecole des Ponts et Chausées.

The Institute of Transportation and Traffic Engineering of the University of California has provided me with excellent opportunity for this work, and I am indebted to Professor Harmer Davis and the late Professor J. H. Mathewson for creating an atmosphere so conducive to research. The manuscript was very efficiently typed by Allan S. Jacobson.

I am most particularly grateful to Richard Bellman. He suggested that I write the book and frequently gave me a good clue to some mathematical difficulty.

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Contents

Preface v

Chapter 1

Probability and Statistics

1.1 Introduction 1
1.2 Discrete Distributions 4
1.3 Random Points on a Line 8
1.4 Continuous Distributions 9
1.5 Mixed Distributions 13
1.6 Cumulative Distributions 15
1.7 Moments and Expectation 19
1.8 Bivariate Distributions 23
1.9 Auxiliary Functions 26
1.10 Distribution of Independent Sums 28
1.11 Comment 31

Chapter 2

Theory of Queues

2.1 Stochastic Processes 35
2.2 Fundamental Definitions 36
2.3 Random Arrivals and Departures 38
2.4 Random Arrivals 45
2.5 Random Arrivals; Regular Service 51
2.6 Busy Period 53
2.7 Mean Values 58
2.8 Generalizations 60
2.9 Comment 62

Chapter 3

Fundamental Characteristics of Road Traffic

3.1 Introduction 67
3.2 The Fundamental Diagram of Road Traffic 69
3.3 Statistical Deduction of Fundamental Diagram 73
3.4 Car Following Deduction of Fundamental Diagram 76
3.5 Fluid Analogy Deduction of Fundamental Diagram 79
3.6 Experimentally Obtained Forms for Fundamental Diagram . . 80
3.7 Comparison of Flow Density Relationships 81
3.8 Car Following for Cases IV, V, VI, and VII 85
3.9 A Fifth Boundary Condition 87
3.10 Comment 89

Chapter 4

Arrangement of Cars on a Road

4.1 Introduction 97
4.2 Starting Density 98
4.3 Gap-Counting Relationships 100
4.4 Erlang Process for Headway 103
4.5 Translated Distributions for Spacing 107
4.6 Compound Gap Distributions 111
4.7 Velocity Distributions in Space and Time 114
4.8 Asymptotic Results 117
4.9 Renewal Theory 123
4.10 Continuous Traffic 125
4.11 Dependent Gaps 127
4.12 Comment 129

Chapter 5

The Simple Delay Problem

5.1 Introduction 135
5.2 Pedestrians Delayed by General Traffic 136
5.3 Pedestrians Delayed by Random Traffic 139
5.4 Randomly Arriving Pedestrians 141
5.5 Blocks and Unblocks 142
5.6 Delayed Vehicles; General Formulation 144

5.7 Delayed Vehicles; Special Cases 150
5.8 Delayed Vehicles; Integral Equation Method 152
5.9 Delayed Vehicles; Bulk Service Model 157
5.10 Delayed Vehicles; Discrete Time Model 161
5.11 Comment 162

Chapter 6

Miscellaneous Problems

6.1 Introduction 167
6.2 Choice of Traffic Light Phases 167
6.3 Left Turning 170
6.4 Multilaned Highways 173
6.5 Sequences of Traffic Controls 176
6.6 Parking 180
6.7 Merging 184
6.8 Bottleneck 188
6.9 The Rush Hour 192
6.10 Comment 195

Chapter 7

A Two Lane Road

7.1 Introduction 199
7.2 Various Speed Distributions 201
7.3 Typical Cycle for a Vehicle 205
7.4 Zero-One Model 209
7.5 Isoveloxic Traffic 211
7.6 Distributed Speeds; Following Time Conjectures 215
7.7 Queueing Model 220
7.8 Comment 223

Supplementary References 227

Author Index 235

Subject Index 239

CHAPTER 1

Probability and Statistics

1.1 Introduction

Experiments can be classified into one of two categories, depending on whether their outcomes are certain or uncertain. A *certain experiment* will yield exactly the same value whenever the experiment is repeated under the same conditions; an *uncertain experiment* will give a variety of values. The distinction is pragmatic rather than logical, for many certain experiments can be made uncertain by various methods, particularly refinement of instrumentation. The repeated measurement of length of a line may give the same value if the ruler is coarsely calibrated and several different values if the ruler is more finely calibrated.

Nevertheless, the distinction between certainty and uncertainty is a useful concept, and the mathematical form of the result is quite different. In a certain experiment, the result will consist of a single constant for each quantity measured. The result of an uncertain experiment is called a *random variable*; the study of uncertain results is the science of statistics. A random variable is completely described not by a number but by a function† which shows the relative frequency (empirical or theoretical) of occurrence of particular values. The function is called a *statistical distribution*.

Consider the simple experiment of measuring the elapsed time between the arrival of the front bumpers of consecutive cars in a single lane of traffic. If the cars are rigidly scheduled, each value will be exactly the same, and the experiment will not be statistical.

† As we shall see there are several functions which are essentially equivalent.

However, in an actual traffic situation, many different numbers will be obtained, and these correspond to a statistical distribution. It is an important problem to determine a suitable mathematical function which will accurately describe the relative frequency of different headways, and so characterize the flow of traffic.

If an experiment is uncertain, it is necessary at first to decide what category of results are possible. The region of possibility may form a set of numbers, called the domain of definition for the distribution.

EXAMPLE 1. Number of kittens born in a litter. Domain of definition: 1, 2, 3, 4, ...

EXAMPLE 2. Number of cars counted in a minute of observation. Domain of definition: 0, 1, 2, 3, ...

EXAMPLE 3. Age of an individual. Domain of definition: $(0, \infty)$.

EXAMPLE 4. Number shown on a throw of a die. Domain of definition: 1, 2, 3, 4, 5, 6.

EXAMPLE 5. Spacing between consecutive cars, front bumper to back bumper. Domain of definition: $(0, \infty)$.

EXAMPLE 6. Spacing between consecutive cars, front bumper to front bumper. Domain of definition: (Δ, ∞) where Δ is a car length.

In all the examples given above, with the exception of Example 4, the upper limit is infinite. This may seem wrong in some cases, but it should be remembered that the domain of definition shows possible values, not probable values. In Example 1, values such as -2 or $\frac{1}{2}$ are theoretically impossible but 10, 20, or larger values are simply improbable. It could be argued that by taking a much bigger integer, a truly impossible value could be found. The difficulty in this case is knowing where to truncate. It is more convenient mathematically to adopt the fiction that (perhaps with probability 10^{-1000}) 250 kittens could be born in a litter than to say that it is theoretically possible that 249 could, but theoretically impossible that 250 could. The choice of a convenient domain of definition is perhaps a question of judgment, which can be gradually acquired, as much as a matter of pure logic.

Another aspect of the examples above is the distinction between discrete and continuous domains. In Examples 1, 2, and 4, the only admissible values are positive integers, while in the other cases, a whole range of real values are permitted. This distinction is easier to make than that between finite and infinite domain but even here there may be some doubt. If a measuring instrument is calibrated only in inches, the results it gives would be discrete. As the calibration is improved, the space between possible values decreases, and it is not difficult to see that if there is such a thing as a "true" value, it could be any real positive number.

It turns out that virtually every reasonable experiment which is wholly continuous or wholly discrete will have one of the following domains of definition:

A. The whole real line, $(-\infty, \infty)$. For example, the error in a measurement of length.

B. The positive half of the real line, $(0, \infty)$. For example the age of an individual.

C. A piece of the real line, (a, b). For example, the angle a thrown needle makes with a fixed line on the floor, $a = 0$, $b = \pi$.

D. All the positive integers, beginning with some value, n, $n + 1$, $n + 2, \ldots$. In Examples 1 and 4, $n = 1$, and in Example 2, $n = 0$.

E. Some of the positive integers. In Example 4, we have just six of the integers possible.

In addition to discrete and continuous domains of definition, there is a third type of domain which is important in road traffic theory: the mixed discrete and continuous. As an example of this, imagine a highway on which all private cars obeyed a speed limit which was given as: minimum 30, maximum 60; and suppose that tractors were allowed to use the road if they traveled at exactly 10 m.p.h. Then the domain of definition would be the continuous interval (30, 60) plus the single value 10.

In the next four sections, we discuss continuous, discrete, and mixed probability distributions, with special emphasis on those important for road traffic.

1.2 Discrete Distributions

Suppose we have some set of the positive integers which are possible outcomes of an experiment. A complete statistical description of that experiment is given when the probability of each outcome is specified. In some cases, such as throwing fair dice, these probabilities can be deduced from the specifications of the experiment. In other cases, such as the birth of kittens, the probabilities can only be determined empirically. Even in the latter cases, however, it is sensible to deal with theoretical values for the probabilities, so that hypothetical functions can be compared with experimental results.

A set of numbers qualifies as probabilities for all the possible results of an experiment provided they are (1) never negative, and (2) add up to unity. Thus the figure zero is taken as the smallest allowable value for a probability, and one as the largest, roughly corresponding to impossibility and certainty.

Let N denote the outcome† of the experiment, where $N = 0, 1, 2, \ldots$. Then we write for the probability that N has the value n

$$\text{Prob}\,(N = n) = p_n$$

where $0 \leqslant p_n \leqslant 1$, and $p_0 + p_1 + p_2 + \ldots = 1$.

EXAMPLE 1. Observations on cat life show the table (see facing page) for the frequency of litters of various sizes.

Dividing by 149, we obtain an *empirical* probability distribution for this particular breed of cats (see below table).

† It is now standard practice to denote a random variable by a capital letter and the corresponding *dummy variable* in its probability distribution function by the corresponding small letter. In the continuous and mixed cases the letters X, x are frequently used and in the discrete case the letters N, n. However, once in this section (Binomial distribution) and once in the next it will be convenient to call a certain constant N.

Number of kittens	Number of litters
1	3
2	7
3	17
4	25
5	33
6	29
7	20
8	9
9	4
10	2
>10	0
	149

$$p_1 = 3/149 \qquad p_6 = 29/149$$
$$p_2 = 7/149 \qquad p_7 = 20/149$$
$$p_3 = 17/149 \qquad p_8 = 9/149$$
$$p_4 = 25/149 \qquad p_9 = 4/149$$
$$p_5 = 33/149 \qquad p_{10} = 2/149$$
$$p_n = 0, \qquad n > 10.$$

In the remainder of this book, probability distributions will be theoretical, rather than empirical. Let us consider, therefore, some probability distributions which can be found by reasoning, without recourse to trial.

If a fair die is thrown, the probability distribution is $p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = 1/6$. How is this known? It might be mistakenly assumed that the values 1/6 could only be determined by experiment. Actually, the situation is quite the reverse. The list of six 1/6's is in fact the *definition* of a *fair* die. The statistical problem then becomes one of deciding whether or not any particular die is fair.

Any convergent series of positive constants can be used to form a probability distribution. We define three of these which are of importance in road traffic theory, and then show the types of traffic experiments which yield these distributions.

GEOMETRIC DISTRIBUTION. Consider the geometric series

$$1 + \rho + \rho^2 + \rho^3 + \dots .$$

If $\rho < 1$, this is a convergent series, and has the sum $1/(1 - \rho)$. Therefore, a valid probability distribution can be defined simply by multiplying each term of the series by $(1 - \rho)$, as follows:

$$\begin{aligned} p_0 &= (1 - \rho) \\ p_1 &= (1 - \rho)\,\rho \\ p_2 &= (1 - \rho)\,\rho^2 \\ &\;\;\vdots \end{aligned} \qquad (1)$$

with the general expression $p_n = (1 - \rho)\rho^n$, $n = 0, 1, 2, \dots$. In Chapter 2, we shall see that an important quantity, the length of a queue, has this distribution.

POISSON DISTRIBUTION. Consider the exponential series

$$1 + \lambda + \lambda^2/2! + \lambda^3/3! + \dots ,$$

which is convergent for all λ, and has the sum e^λ. Dividing by e^λ, we obtain a valid probability distribution, defined as follows:

$$\begin{aligned} p_0 &= e^{-\lambda} \\ p_1 &= \lambda\, e^{-\lambda} \\ p_2 &= \lambda^2\, e^{-\lambda}/2! \\ &\;\;\vdots \end{aligned} \qquad (2)$$

with the general expression $p_n = \lambda^n\, e^{-\lambda}/n!$, $n = 0, 1, 2, \dots$. This distribution, as we shall prove later, describes the probability that exactly n randomly† arranged cars will be observed in unit length of road.

† The word *random* is used in two ways in statistics: (a) as the equivalent of *uncertain*, and (b) as the equivalent of *Poisson*. In this book it will always have the second meaning, except when speaking of a *random variable*.

BINOMIAL DISTRIBUTION. Suppose we choose two numbers which add up to unity, say $p + q = 1$. Expand $(p + q)^N$ in a binomial series, where N is a fixed integer, and obtain

$$p^N + p^{N-1}q + \binom{N}{2}p^{N-2}q^2 + \binom{N}{3}p^{N-3}q^3 + \dots + pq^{N-1} + q^N = 1.$$

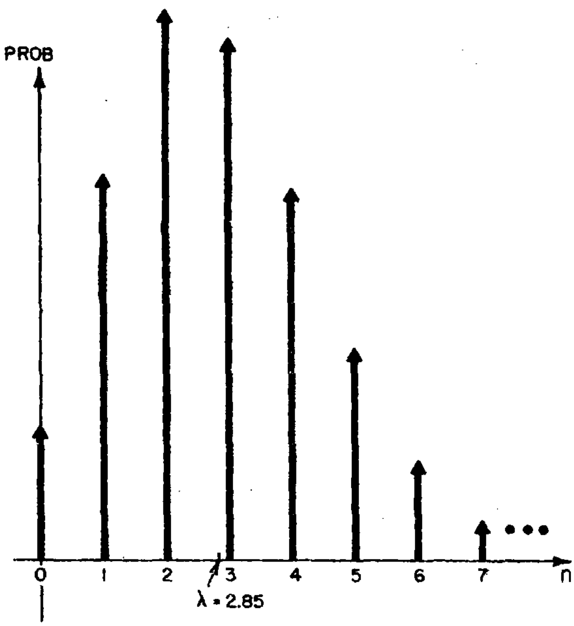

FIG. 1. A discrete distribution.

Therefore, a valid probability distribution can be defined as

$$p_n = \binom{N}{n}p^{N-n}q^n, \qquad n = 0, 1, \dots, N.$$

Notice that for this distribution, the domain of definition consists of only $N + 1$ values.

It is possible to represent a discrete probability distribution geometrically by erecting an ordinate of length p_n over each value n. Figure 1 shows a typical form for the Poisson distribution.

1.3 Random Points on a Line

A Bernoulli experiment is an experiment which has only two possible outcomes, with constant probabilities p and $q = 1 - p$; for example, the flipping of a coin. For convenience, the two outcomes are called "success" and "failure." If a Bernoulli experiment is performed N times in succession, the probability of exactly n successes ($n = 0, 1, \ldots, N$) is the binomial distribution.

Proof. The probability of getting first n successes is p^n, and then $N - n$ failures is q^{N-n}; therefore, the probability of n successes followed by $N - n$ failures is $p^n q^{N-n}$. But a total of n successes may be arranged by permuting the above sequence in any one of the $\binom{N}{n}$ possible ways, leading to the binomial expression.

Example. A fair coin is tossed 50 times. What is the probability of exactly 25 heads? $p = q = \frac{1}{2}$, since the coin is fair, $N = 50$, $n = 25$, and therefore

$$\text{Prob (25 heads)} = p_{25} = \binom{50}{25}\left(\frac{1}{2}\right)^{25}\left(\frac{1}{2}\right)^{25}.$$

Now suppose that points are dropped at random on an infinitely long line in such a way that the average number of points per unit length is λ. Then the probability of finding exactly n points on a finite segment of length τ is the Poisson probability with λ replaced by $\lambda\tau$.

Proof. Take a finite segment of length t which completely surrounds τ. In dropping each of N points, we have a Bernoulli experiment in which the probability of a success (falling on τ) is τ/t, and the probability of failure (falling on t but outside τ) is $1 - \tau/t$. Therefore, by the previous result, the probability of obtaining exactly n points in τ is

$$p_n = \binom{N}{n}(\tau/t)^n (1 - \tau/t)^{N-n}, \qquad n = 0, 1, 2, \ldots, N.$$

Let t approach infinity, and N approach infinity in such a way that N/t approaches the constant density of points λ.