Algebraic Structures, Self-consistent Sources and Conservation Laws
—A Coupled Soliton Equation Hierarchy

（孤子耦合方程族的代数结构、自相溶源和守恒律）

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沈阳师范大学学术文库系列丛书

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Preface

Recently, the chaos, soliton and fractal have important positions in nonlinear science, which have many important applications in some fields such as physics, chemistry, mechanics, biology, etc. Soliton theory and integrable system have penetrated into many disciplines, and many scholars have done a lot of works in this respect. With nonlinear science developing, and the corresponding nonlinear equation is becoming more and more rich contents. As one of the important branches of nonlinear science, soliton theory is widely used in many fields of nonlinear science.

The main contents of the book include the following: In chapter 2, we would like to present a definition of the bi-integrable couplings of continuous and discrete soliton hierarchies, which contain two given integrable equations as their sub-systems. There are much richer mathematical structures behind bi-integrable couplings than scalar integrable equations. And it is shown that such bi-integrable coupling system can possess zero curvature representation and algebraic structure associated with semi-direct sums of Lie algebras. As application examples of the algebraic structure, the bi-integrable coupling system of the MKdV and generalized Toda lattice equation hierarchies are presented from this theory.

In chapter 3, it is shown that the Kronecker product of matrix Lie algebra can be applied to construct a new integrable coupling system and Hamiltonian structures of continuous and discrete soliton hierarchies. Furthermore, we construct the Hamiltonian structure of integrable couplings of soliton hierarchy by using the Kronecker product. The key steps aim at constructing a new Lax pairs by the Kronecker product. As illustrate examples, direct application to the continuous and discrete spectral problems lead to some novel soliton equation hierarchies of integrable coupling system. Then, we present the Hamiltonian structure of integrable couplings of continuous and discrete hierarchies with the component-trace identity.

Soliton equations with self-consistent sources have important physical applications, such as hydrodynamics, solid state physics, plasma physics, and so on. The soliton solutions can explain the interactions between different solitary waves. In particular, AKNS, MKdV and KdV equation hierarchies with self-consistent sources have been presented. Complexiton solutions of the Korteweg-de Vries equation self-consistent sources, soliton and positon solutions of the Schrödinger equation are presented. However, these methods are not designed to construct the integrable coupling equation hierarchies with self-consistent sources. In chapter 4, we present the continuous, discrete and non-spectral integrable couplings of soliton equation hi-

erarchies with self-consistent sources by using of loop algebra $\tilde{sl}(4)$. As application, the generalized Wadati-Konono-Ichikawa equation, discrete equation and other integrable couplings with self-consistent sources are derived through the enlarged loop algebra $\tilde{sl}(4)$.

As is well known, the conservation laws(CLs) play important roles on discussing the integrability for soliton equations. Since MGK's discovery of an infinite number of conservation laws for KdV equation, many methods have been developed to find them. Conservation laws play an important role in mathematics and engineering as well. Many papers dealing with symmetries and conservation laws are presented. In chapter 5, the conservation laws of a nonlinear integrable coupling system are derived for the first time. Beginning with Lax pairs from special non-semisimple matrix Lie algebras, we present a scheme for constructing the nonlinear integrable couplings, and consider the conservation laws of a nonlinear integrable coupling system. We construct a nonlinear integrable couplings of continuous and discrete soliton hierarchies, then the infinite conservation laws for the nonlinear integrable couplings of the equation hierarchies are established.

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Chapter 1

Introduction

Nonlinear science is an important basic science and studys the nonlinear common-phenomenon, which is gradually developed on the basis of integrated disciplines since 1960. And it is known as "The third revolution" of natural science in the last century. Nonlinear science relates to all fields of social sciences, and changes people's traditional views to the real world. Nonlinear science research not only has an important scientific significance, but also has a practical significance for human survival environment. It is well known that the body of the nonlinear science research is Chaos, Solitons and Fractrals. Among them, the representative of soliton in nonlinear science is the unforeseen organized behavior, which is the dispersion and nonlinear dynamic system. The soliton has an important and historic moment in the mathematical theory of the algebra, geometry and topology. Therefore, the research on the soliton theory is an important issue of mathematical physics, which is the frontier of nonlinear science.

1.1 Discovery and development of the soliton

Soliton phenomenon was first discovered by the British physicist Scott Russell[1] in 1834. He gave a report " Wave Theory" at science meeting of British association in September 1844, and he found an invariable water body in the canal and the water mass disappeared in a couple of miles away in river bend in August 1834. In order to study the kind of phenomenon more carefully, Russell made a lot of experiments in the laboratory, and found that the solitary wave has the character of shallow long nature.

Airy, Stokes, Boussinesq[2] and Rayleigh presented the further research for this wave. In order to approximately describe the solitary wave, Boussinesq presented a 1-dimensional nonlinear evolution equation, namely the Boussinesq equation. But the work is still not make the solitary wave scientists completely convinced. Especially, is there kind of water wave equation for Russell and others observed solitary wave? It has been bothering people for a long time.

Korteweg and his doctoral de Vries[3] have presented a nonlinear evolution equa-

tion (KdV equation) until 1895. They used a solitary wave solution of the equation to explain the Russell observed shallow water wave, which is theoretically proved the existence of solitary wave solution. However, is this wave stable after the collision of two wave deformation? These problems have not been solved for a long time. So much so that some people doubt, since the KdV equation is nonlinear partial differential equation, the solution of the superposition principle is no longer after the collision. The shape of solution is likely to damage. Hold this view believe that the "wave" is "unstable", so it has no physical meaning, then the study of solitary wave is to run aground.

In 1955, the famous physicists Fermi, Pasta and Ulam[4] presented the famous FPU problem with about 64 particles of nonlinear spring connected into a nonlinear vibrating string. Initially, the all energy of harmonic oscillator is focused on a particle, namely the initial energy of other 63 particles are zero. After a long time, almost all of the energy back to the original initial distribution. This is contradiction with the classic theory. At the time, because only considering the problem of the frequency space, the solitary wave solution was not found, which failed to properly explain the problem.

In 1962, Perring and Skyrme[5] presented the numerical experiment of sine-Gordon equatioin on study of basic particle model. The results showed that the solitary wave solution of this equation keep its original shape and speed after collision. In order to explain the phenomenon in the FPU problems, Kruskal and Zabusky[6] considered the FPU problems from the viewpoint of continuum in 1965. In the continuous case, the FPU problems approximately could be described by KdV equation. Two different wave velocities of KdV equation were studied, if the two solitons moving apart and wave velocity is big on the left, after the collision, the big wave velocity is the right and keeps the original height and speed. The collision of two solitons is elastic collision, and is similar to the particle, so they called it the soliton. Soliton is also referred to as the solitary wave sometimes, which refers to the features of solution in nonlinear partial differential equations. And the corresponding physical phenomena of these properties are: ① the energy is focused on small area; ② the interaction of two soliton appears elastic scattering phenomenon, namely the wave form and wave velocity can restore to the original, which reveals the essence of the solitary wave accurately. After more than 20 years, soliton theory research work was more vigorous development and had penetrated into many areas, such as the many branches of physics (elementary particle physics, plasma physics, condensed matter physics, fluid, superconducting physics, laser physics, biological physics, etc.), biology, optics, astronomy[7-16]. The concept of soliton is a great progress in the field of applied mathematics in recent years. Soliton theory

can be roughly divided into two aspects, one is developing a method to solve non-linear equation system, another is using a series of wonderful algebra and geometry properties to study the integrability.

1.2 Development situation of integrable system

Integrable system is one of the main content of the contemporary nonlinear science, it has experienced a long historical development process. Newton was the first scientist who studied integrable system. When he studied two celestial bodies under the action of gravity movement, and discovered that the calculus is a basic tool, and established a differential equation of celestial body movement, through solving the differential equation, he got the Kepler's 3-laws of planetary motion. This is the most brilliant work for Newton invented calculus. And two-body problem was regarded as a model of quadrature method. People started to solve the motion differential equation as the main work in mathematical physics for a long time, and found a series of integrable examples, such as: Jacobi ellipsoid geodesic flow[17], citigroupeumann constraints on the spherical harmonic oscillator[18], Euler, Lagrange, Kovalevski integrable gyro[19, 20], and so on, which are powerful methods for the techniques of integrability.

In addition to a number of simple elementary integrable types of differential equations, the discovery of new integrable model is more and more difficult, and requires a great deal of mathematical skills. Poincare finally realized that the Hamiltonian system is not integrable until the 19th century. Especially in 1887, Brun proved the famous three-body problem is not integrable[21], because it has not enough conservation integral. Further Brun found integrable system is damaged under small disturbance integrability. The importance of such integrability nature is great challenge, and thus be shelved. From the end of 19th century, Poincare and Birkhoff interested in research of power system from integrable theory into qualitative theory, the theory of integrable was the low tide until the sixties of the 20th century.

Until the mid sixties of the 20th century, people found many equations although have different backgrounds, but they are the completely Liouville integrable systems [7, 10, 22]. Under small disturbance, although the completely integrability is damaged, but the original problem remains the same while retaining a large subset of the ring, forming a complex is a measure of Cantor set, this is the famous KAM theory. Someone further evidenced that the perturbation system was still integrable in the Cantor set under the sense of Whitney.

Theoretical framework is symplectic manifold for Hamiltonian system, the theory of finite dimensional symplectic manifold was introduced in Arnold's book [23], the infinite dimensional symplectic manifold has not completely presented. For $2N$

Hamiltonian system, the Liouville integrable is to find N independent and conservation of two integral. When the conservation level set is an invariant submanifold for each integral conservation integral manifold, if the level set is tight and connected, it is differential homeomorphism with $N - d$ torus. The Hamiltonian system is given rise to as following

$$\dot{I} = 0, \quad \dot{\phi} = \omega(I),$$

and it gives rise to the solution

$$I(t) = I(0), \quad \phi = \phi(0) + \omega t,$$

according to the phase space coordinate inversion, it can display solution of original problem, which is the theory of Liouville-Arnold [23].

So far, there is not overall grasp completely integrability of infinite dimensional Hamiltonian system, only some local properties are studied, usually it takes the Lax integrable or Liouville integrable to make judgement for integrability on nonlinear evolution equations. The Lax sense of integrable requires that a nonlinear evolution equation has satisfied the Lax or zero curvature represention. The Liouville integrable requires that the nonlinear evolution equation can be written as the generalized Hamilton equation, and there are countable and two conservation densities. Magri, Vinogradov, Olver, and others successively presented the classical Hamiltonian system of Lie on the theoretical frame work based on manifold-Possion or Possion brackets [24-26]. There is not the limit to the dimension of Poisson flow, which can be any finite or infinite dimension, the generalized Hamilton can form Hamiltonian system of evolution equations. Lie-Possion structure has been successfully applied to free fixed-point rotation of rigid body Euler equation [27-29].

At present, the study of integrability for nonlinear evolution equation mainly concentrated in two aspects: how to determine the integrability of the equation, and how to establish a better integrable model from mathematics theory, especially the infinite dimensional Hamiltonian system. Many scholars made a lot of fruitful works in this field. In 1975, Wahlquist and Estahrook proposed the prolong structure of equation by using Lie algebraic structure, this method can effectively construct Lax pairs of equation.

Two kinds of integrability definitions are given for infinite dimensional Hamiltonian integrable system [13], the Lax integrable and Louville integrable. The important work in integrable system is as following: ① Given a nonlinear evolution equation, whether is it Lax integrable; ② To find many integrable systems, and generate significant nonlinear evolution equations.

In order to obtain a Lax integrability of nonlinear evolution equations, which is searching for Lax pairs of zero curvature, the prolong structure method is more

successful. In 1983, Drinfeld and Sokolov constructed the Lax representation of KdV equation with Kac-Moody algebra. Date developed τ-function method. In 1985, Gu Chaohao and Hu Hesheng presented a class criterion of equation integrability, based on the curved surface of the basic equation, which is one of the important progress in this field [13]. Cao Cewen presented invariant spectrum method and a new transposition framework in 1989 [30].

Tu Guizhang, Boiti, Panpirelli etc. presented a simple way to search for Hamiltonian structure of soliton equation since 1982 [31]. In 1988, Tu Guizhang proposed the famous trace identity by using variational technique [32, 33], and constructed some Hamiltonian structures of soliton equations. Ma Wenxiu called the format on "Tu format", and established a number of integrable Hamiltonian systems [34, 35]. Hu Xingbiao studied the Tu format with the Loop algebra $\tilde{A}_1$ and $\tilde{A}_2$, then presented the expanding representation of the trace identity [36]. Fan Engui took Tu format and the Darboux transformation, and made many meaningful works in soliton theory [37].

Fuchssteiner, Fokas and Anderson presented an approach to research Hamiltonian structure [38, 39]. In this method, the recursion operator plays a key role. With symmetry properties of recursion operator L, one of the conditions is the Hamiltonian structure: L is an inverse-symplectic decomposition. Chen Dengyuan, Zhang Dajun improved the conclusion of Fuchssteiner, Fokas and Anderson, and used the theory implicitly to prove that $1 + 1$ Lax integrable system of recursion, such as the relevant results of spectrum equation were extended to the discrete soliton system [40].

Another important feature is the infinite conservation law of soliton equation. Since Grdner and Kruskal found infinite conservation law of KdV equation, a series of methods have been constructed, including Wadati etc made considerable contributions [41, 42]. In 1998, Wadati and Tsuchida gave the multiple system of conservation law with important trace identity. Chen Dengyuan, Zhang Dajun and Zhu Zuonong presented a method to construct Lax conservation law of discrete systems [43, 44].

From high dimensional integrable system to generate low dimensional integrable system is an important way of expanding integrable system. Since 1989, Cao Cewen, Geng Xianguo presented the relation between the potential function and characteristic function, and produced the Lax nonlineartion of nonlinear finite dimensional integrable Hamilton [45]. Then, they had given the involution solution of the soliton equation [46]. Zeng Yunbo, Li Yishen developed the nonlinear method, and presented the condition of high order symmetry constraint, derived a finite dimensional Hamiltonian system from an infinite dimensional Hamiltonian system, which is de-

composed into interchangeable finite dimensional Hamiltonian system with x and t_n in a zero curvature framework[47-51]. Ma Wenxiu et al considered a single nonlinear bi-nonlineartion method through the spectrum problem of Lax pairs, obtained finite dimensional Hamiltonian structure and symmetry constraint flow[52-55]. Ma Wenxiu, Li Yishen studied the asymmetric flow constraints[56]. Cheng Yi, Li Yishen, Zhang Youjin constructed KP integrable system based on the theory of the Sato, quasi differential operator and K constraints, including the AKNS system[57-59], Chen Dengyuan considered K-constraint of the MKP system, obtained low dimensional nonlinear Schrödinger system. Zhu Ningsheng, Zhu Min proved symmetric and asymmetric constraints by using Gateaux derivative and the functional derivative[60]. Chen Dengyuan, Zeng Yunbo, Li Yishen also introduced the conversion operator, and derived the equivalent relations between some equations[61-63].

Recently, the spectral gradient method, Tu format method and other methods are used to study integrability. Fuchssteiner presented the concept of "Integrable couplings", which is originated in Virasoro symmetrical algebra of the integrable system without a central element. Ma Wenxiu presented a perturbation method to generalize integrable couplings, and studied the integrable coupling system of KdV equation. Guo Fukui, Zhang Yufeng and Xia Tiecheng constructed a new method and obtained some new integrable coupling systems.

1.3   Development of exact solution in nonlinear evolution equation

The study of a lot of problems divided into two categories in the natural sciences: one is qualitative research; another is quantitative research. The quantitative study can be subdivided into numerical approximate and precise structural research. People have gotten a lot of meaningful nonlinear evolution equations since KdV equation had been proposed in 1895. But because of the complexity of the differential equation, a lot of important equations can't be found the exact solution. Besides, the physical meaning of the data processing has yet to be further constructed and found. Fortunately, the mathematicians and physicists discovered a series of effective methods to construct exact solutions in soliton theory, such as Bäcklund transformation, Darboux transformation, similar transformation, inverse scattering method, Painlevé truncation, Hirota bilinear method, variable separation method, function expansion method, and so on.

The Swedish geometrician Bäcklund[64] found the following relation between the solutions u and u' of sine-Gordon equation $u_{\xi\eta} = \sin u$ as following

$$u'_\xi = u_\xi - 2\beta \sin\left(\frac{u+u'}{2}\right), \quad u'_\eta = -u_\eta + \frac{2}{\beta}\sin\left(\frac{u-u'}{2}\right), \qquad (1.3.1)$$

when they studied the negative constant of surface, which is called Bäcklund transformation. In 1882, Darboux[65] studied an eigenvalue problem of one-dimentional Schrödinger equation($\lambda_t = 0$)

$$- \phi_{xx} - u(x,t)\phi = \lambda\phi, \tag{1.3.2}$$

and found that: Let u and ϕ satify(1.3.2), considering a constant λ_0, set $f(x) = \phi(x, \lambda_0)$, ie. f is a solution for (1.3.2) and $\lambda = \lambda_0$, according to

$$u' = u + 2(\ln f)_{xx}, \quad \phi'(x, \lambda) = \phi_x(x, \lambda) - (\partial_x \ln f)\phi(x, \lambda), \quad f \neq 0, \tag{1.3.3}$$

the functions u', ϕ' satify equation(1.3.2), the transformation(1.3.3) is called Darboux transformation. The basic idea of Darboux transformation is: From a solution and its Lax solution of nonlinear equation, it obtains the new solution and Lax solution by using the algebra algorithm and differential operation. Sometimes people called Darboux transformation is also known as Bäcklund transformation, or called "Darboux method of Bäcklund transformation".

The Norwegian mathematician Sophus Lie[66] inspired by Abel and Galois dealing with algebra equations and introduced the concept of continuous group, then he presented a variety of different methods to solve ordinary differential equation during the late 19th century. This kind of continuous group became known as the Lie group, also known as the same group or symmetry group. Lie proved that: if a differential equation in the transformation of single parameter under the action of Lie groups remains the same, it can be reduced one order. Lie's work comprehensively elaborated many topics, including integrating factor, homogeneous equation, the linear equation, Euler equation and the Laplace transformation. The partial differential equation, the linear partial differential equation has invariance under the action of Lie group, through the transformation can directly lead to the stack solution.

Lie had established the heat conduction equation of local transformation group[67], and taken Lie group theory to the partial differential equation. In 1905, the French mathematician Poincare proved that Lorentz transformation can constitute the symmetry group of Maxwell's equations, and Maxwell equations remain the transformation of Lie group. In 1958, the mathematicians Ovsiannikov, Venikov Soviet developed excellent work in Lie group. Bluman and Cole [68, 69] expanded the Lie group method and presented the classical Lie group method (i.e., symmetrical) in 1969. Olver [70] proved that how to obtain an infinite number of partial differential equations for symmetric in 1977. Olver [71] found the solution u of KdV equation and the two symmetrical independent variables (x, t) in 1980. In 1989, Clarkson and Kruskal[72] proposed the direct method to reduce differential equation, this method

is not used as group theory. The method directly applied in the Boussinesq equation, and found a new symmetric, which is the classical Lie group method just a special case of a direct method. Lie symmetry theory has a great and profound influence in mathematics, mechanics, physics, engineering and other modern sciences.

Gardner, Greene, Kruskal and Miura(GGKM) [73] used the inverse scattering problem of Schrödinger equation, and transfored the initial value problems of KdV equation into three solving problems of linear equations, then obtained N soliton solutions of KdV equation, which is called the inverse scattering method. In 1968, Lax [74] proposed GGKM to solve KdV equation by using inverse scattering method and presented a more general framework of other partial differential equation, and pointed out Lax represention by inverse scattering method [6]. Thus an example is used to prove the general inverse scattering method.

In 1971, Hirota [16] introduced the bilinear method and used to construct many soliton solutions and Bäcklund transformation. Recently, Hu Xingbiao et al [75,76] developed the method and presented reciprocal theorem of solutions. In 1988, Boiti et al [77] used this method to study the (2+1)-dimensional model, and presented a special soliton solution-dromion structure. In 1993, Rosenau and Hyman [78] studied the effect of nonlinear dispersion model and considered the $K(m,n)$ model, then gave the compacton solution, which has the interesting elastic collision and is similar to the soliton solution.

Because of the complexity of the nonlinear evolution equation itself, the existing methods may not be able to find out the nontrivial solutions and exact solutions of some equations. And for the same kind of equation, the method may be different, so there is not any kind of method can contain other methods, unless this method can work out all the solutions of the equation. This would require to find more effective methods to research the solutions of differential equation.

Chapter 2

Algebraic Structure of a Coupled Soliton Equation Hierarchy

Recently, the spectral gradient method, "Tu method" and other methods study the integrability of soliton equation hierarchy. Fuchssteiner presents the concept of "integrable coupling", and it originates in the integrable system without a central element of Virasoro symmetry algebra. Ma Wenxiu uses the perturbation method and presents a generalized theory to study the KdV integrable coupling system, but this method is more troublesome, the generalization is not strong. Guo Fukui and Zhang Yufeng use the expnding Lie alggebra method to obtain new integrable coupling system. Inspired by those works, we consider the Lax pairs and propose the spectral expansion method, which can obtain many new integrable coupling equations. This method is extended to high dimension space, and obtain a series of multi-component integrable coupling of soliton equation. We can give the generalized killing product, and obtain the Hamiltonian structure of multi-component soliton hierarchy by quadirc trace identity.

2.1 Kac-Moody algebra

In 1948, C. Chevalley published an article on "Rendus Comptes". This article not only contains a unified algebraic theorem of E.Cartan about the representation of a finite dimensional complex semi-simple Lie algebra, but also presents a number of important concepts. These concepts are the basic concepts of Kac-Moody algebra. In particular, he proved that the finite dimensional complex semi-simple Lie algebra g of any n rank has $3n$ generating elements $e_1, \cdots, e_n, f_1, \cdots, f_n, a_1^\vee, \cdots, a_n^\vee$, and they satify the following relations:

$$[a_i^\vee, a_j^\vee] = 0, \quad i,j = 1,2,\cdots,n,$$

$$[e_i, f_j] = \delta_{ij} a_i^\vee, \quad i,j = 1,2,\cdots,n,$$

$$[a_i^\vee, e_j] = a_{ij} e_j, [a_i^\vee, f_j] = -a_{ij} f_j, \quad i,j = 1,2,\cdots,n,$$

$$(ade_i)^{1-a_{ij}} e_j = 0, (adf_i)^{1-a_{ij}} f_j = 0, \quad i \neq j,$$