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THE THEORY OF INTEGRATION

BY

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PREFACE

THE object of small treatises of this kind is to enable the general student to gain rapid access to the various branches of Modern Mathematics, thereby preventing this science from breaking up into a number of disconnected parts, each belonging to its own specialist and closed to the outsider. Mathematics must form a whole, any progress in one of its parts stimulating advance in the others and raising new problems; when a branch is severed from the tree, it dies.

In writing this book, I have therefore tried above all to simplify the work of the student. On the one hand, practically no knowledge is assumed (merely what concerns existence of real numbers and their symbolism); on the other hand, the ideas of Cauchy, Riemann, Darboux, Weierstrass, familiar to the reader who is acquainted with the elementary theory, are used as much as possible.

I have also hopes that it will be of some use to the initiated, who may find here new points of view and greater generality in the treatment, owing to the idea of integration with respect to a function in space of n-dimensions. I have not however included what Hobson and others call the Fundamental Theorem of the Integral Calculus, namely the connection with the Theory of Derivation.

The Theory of Integration, which forms the subject of this book, has long been one of the most useful tools of Mathematics. Its methods were already employed with success by the ancient Greeks, in their investigations about Areas and Volumes. They possessed the method of exhaustion, the method of series. They were very clear about the idea of limit and this perhaps made them suspicious of the unsound method of infinitesimals, as results thus obtained were always established independently.

After the Dark Ages the rediscovery of this last method and the use of the symbolism of Algebra rendered possible the creation of the Calculus by Newton and Leibnitz. Unfortunately, *believing they had reduced everything to symbols*, they did not realise the need of examining the *ideas* these represented and testing their soundness. They conceived their Calculus to be purely formal; limiting process, a mere operation on symbols. They were very clear as to the properties they expected of such operations: possibility of operation, existence of inverse operation, reversibility of order of two consecutive operations. But they were not so clear as to the properties implied of the entities operated on, which in their case were *functions*, loosely defined as numbers depending on variable quantities.

This conception of Mathematics persisted for a long time. In the nineteenth century, however, the feeling that Mathematics is not the entire property of the mere Calculator, but rather that of the Thinker, revived at last. The result was the development of Geometry, the Theory of Groups, the Theory of Vectors, the systematic use of the Imaginary.

The Mathematicians of that century naturally also perceived the need of reforming the Infinitesimal Calculus. The reform was started with Cauchy's Theory of Limits, based on Inequalities. Cauchy also introduced the notion of *Continuity* and attempted to use it as a foundation for the Calculus. He saw the unsatisfactoriness of the notion, hitherto adopted, of Integration as Inverse Differentiation: a definition which is not constructive requires a theorem ensuring the existence and unicity of the entity in question. Cauchy defined Integration for a continuous function by an always possible limiting process and he proved that it could be considered as the inverse of Differentiation.

But the occurrence of discontinuous functions in certain simple problems and the discovery, by Weierstrass, of continuous non-differentiable functions,—by Riemann, of discontinuous integrable functions, showed that continuity is both inconvenient and unnatural for the foundation of the Calculus.

The Theory of Integration and that of Differentiation have since been built up separately as parts of the New Calculus, the *Calculus of Real Functions*, whose great generality, far from being due to a love of complication on the part of its founders, as was at one time asserted, is to be attributed to the simplicity and straightforwardness of its methods

This New Calculus would never have been possible but for the wonderful ideas of Cantor, at first completely unintelligible to the Mathematicians of the Older School, some of whom even wilfully misunderstood them and sought to lead others into error with regard to them; but which, fortunately, very much influenced a few younger men since become famous. Only too often have ideas of the greatest value been left for a long time unheeded, while their authors remain in obscurity. Galois, the founder of the Theory of Groups, was ploughed at the entrance examination of the École Polytechnique through knowing more than his examiners, and this was only the first of a series of disappointments which embittered his short life. Grassmann, the creator of the theory of vectors, remained a schoolmaster most of his life, and his book, the "Ausdehnungslehre," was burned by the publishers, who could find no buyers.

Cantor's Theory of Sets of Points and Borel's improvement of the theory of content or measure of such sets paved the way to the semi-geometric definitions of Integration, given almost simultaneously by Lebesgue and Young. These definitions represent an extension comparable to that of Arithmetic on the introduction of irrational numbers. They are substantially equivalent to the more direct one here adopted, later given by Young, using the work of Baire on functions.

My father had long thought of writing a connected account of his theory. In carrying out this task myself at his suggestion, I have tried to do justice to his ideas and to introduce a few minor improvements of my own. If I have succeeded in my endeavours, it will have been largely owing to his encouragement, and to the constant assistance of my mother and of my sister Miss R. C. H. Young.

L. C. Y.

September 1926.

CONTENTS

CHAPTER I

THE METHOD OF MONOTONE SEQUENCES

§ 1. Successions of numbers*.

A set of numbers is said to be *bounded* if there exist a finite number greater and a finite number less than all those of the set. The smallest number such that no number greater than it belongs to the set is called the *upper bound* of the set of numbers; the greatest number such that no smaller number belongs to the set is called the *lower bound* of the set. If there is no finite number greater than all those of the set, the set of numbers is said to be *unbounded above*, and we shall agree to say that its upper bound is $+\infty$; similarly, if there is no finite number less than all those of the set, the set of numbers is said to be *unbounded below*, and its lower bound will be $-\infty$.

A countably infinite set of numbers, written down in a definite order, is called a *succession*, e.g.

$$a_1, a_2, \ldots, a_n, \ldots.$$

Repetition of a number is allowed.

A succession is said to be *monotone ascending* if each term is greater than or equal to the preceding,

$$a_1 \leqslant a_2 \leqslant \ldots \leqslant a_n \leqslant \ldots;$$

a succession is said to be *monotone descending* if each term is less than or equal to the preceding,

$$a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n \geqslant \ldots.$$

In either case the succession is said to be monotone, or to be a *monotone sequence*. One of the bounds of a monotone sequence is clearly its first term a_1. The other is called the *unique limit* of the monotone sequence, whether finite or infinite.

Given any succession S

$$a_1, a_2, \ldots, a_n, \ldots,$$

the succession S_r obtained from it by leaving out its r first terms, has its bounds K_r, k_r lying between those of the succession S_{r-1}, i.e. we have

$$k_{r-1} \leqslant k_r \leqslant K_r \leqslant K_{r-1}.$$

Thus, the succession of the upper bounds

$$K_1, K_2, \ldots, K_n, \ldots$$

* We are dealing with real numbers; $+\infty$ and $-\infty$ are regarded as distinct.

is monotone descending. Its unique limit is called the *upper limit* of the succession S. Similarly, the unique limit of the monotone ascending sequence of the lower bounds is called the *lower limit* of S.

The upper and lower limits of a monotone sequence are equal and coincide with its unique limit.

For one succession of bounds coincides with the given sequence and the other consists of terms all equal to the unique limit.

A succession whose upper and lower limits are equal is called a *sequence*, and their common value is called the *unique limit* of the sequence. A sequence having a finite limit is said to converge; otherwise it diverges. A succession which is not a sequence is said to oscillate.

Given any succession S, we call subsuccession of S a succession of numbers all belonging to S and occurring in the same order as in S. We call subsequence of S any subsuccession of S which is a sequence. The unique limit of any subsequence of S is called *a limit* of S.

THEOREM. The upper and lower limits of a succession are limits of that succession.

It is sufficient to prove this for the upper limit.

If the numbers K_n all belong to the succession, the theorem is obvious.

If any one of them does not belong to the succession, all the following are equal to it, and it is the upper limit*. Let r_0 be its index, r_1 that of the first term after a_{r_0} which is greater than a_{r_0}, r_2 that of the first term after a_{r_1} which is greater than a_{r_1}, and so on. Then the subsuccession of S,

$$a_{r_0}, a_{r_1}, \ldots,$$

is a monotone sequence whose unique limit is K_{r_0}, because, by construction, no term of S after a_{r_0} can exceed this limit which, being the upper bound of a subsequence of S_{r_0}, cannot exceed K_{r_0}.

THEOREM. The upper and lower limits of a subsuccession of S lie between those of S.

Let S' be the given subsuccession. Let S_r' be the succession obtained from S' by leaving out its r first terms, and let S_r be obtained similarly from S.

Then S_r' is a subsuccession of S_r, and its bounds K_r' and k_r' lie between those of S_r, K_r, and k_r,

$$k_r \leqslant k_r' \leqslant K_r' \leqslant K_r.$$

This holds for all r; the theorem follows.

* Omitting from a set of numbers a finite number of its elements none of which coincide with the upper bound, does not affect the upper bound.

COROLLARY. All the limits of a succession lie between its upper and its lower limits*.

As a kind of converse of the preceding theorem, we have:

THEOREM. If a finite number of subsuccessions of S together contain all the elements of S, they have among their upper and lower limits those of S.

It is sufficient to prove this for the case of two subsuccessions and to consider only the upper limits. Let U be the upper limit of S. Then there is a subsequence having U as limit and we can so arrange that it belongs entirely to one of our subsuccessions (see footnote*). U is therefore a limit of that subsuccession. By the preceding theorem and its corollary it must therefore be its upper limit.

e-definitions of limits and convergence. To define by this method the upper limit of a bounded succession S,

$$a_1, a_2, \ldots, a_n, \ldots,$$

U is said to be the upper limit if given any positive number e however small, an index N can be found such that from and after $n = N$,

$$a_n \leqslant U + e,$$

while an infinite succession of n's can be found such that

$$a_n \geqslant U - e.$$

Similar definition for lower limit.

Thence the e-definition of limit of a convergent sequence

$$U - e \leqslant a_n \leqslant U + e,$$

for all n from and after N.

This method, which was known to the Greeks, is probably familiar to the student, who will easily prove the equivalence of the definitions so obtained with our former ones.

The characteristic advantage of our method is *to reduce the consideration of all successions to that of monotone sequences.*

§ 2. Successions of functions.

Corresponding to any set of numbers, we have on the straight line a set of points†; we need only choose an origin, a sense, and a unit of length.

We shall say a point is a *limiting point* of our set of points, and that the corresponding number is a limit of the set of numbers, if every interval of which it is the centre contains an infinite number of points of the set. This agrees with our definition of limit in the case of a succession.

* In particular, all the limits of a sequence coincide with its unique limit.

† This is equivalent to what is called the Cantor-Dedekind axiom.

A set containing all its limiting points is said to be *closed*. A point belonging to a set and not a limiting point is called an *isolated point* of that set.

After the finite sets, consisting of a finite number of points, and the sequences and successions of points, the simplest sets are the *intervals*. An interval consists of all the points between its endpoints. If it includes these it is closed, if neither, *open*.

Corresponding to any pair of numbers, we have a point in the plane; corresponding to any set of pairs of numbers, a set of points in the plane. Similarly any set of numbers given n by n, may be taken to represent a set of points *in n dimensions*. We may agree to represent the n coordinates of a point by *a single symbol*, and let x stand for the ensemble of the n numbers $x_1, x_2, \ldots, x_n$.

Two points $a=(a_1, a_2, \ldots, a_n)$ and $b=(b_1, \ldots, b_n)$ such that each coordinate of a is less than the corresponding one of b, define an interval (a, b), consisting of all the points x whose coordinates all lie between the corresponding coordinates of a and b, that is the set of points belonging to the rectangle or to the n-dimensional parallelopiped whose sides are parallel to the axes and which has a and b for its endpoints. The interval is open if it consists only of the internal points, closed, if of all the points. We shall call length of an interval the length of a principal diagonal. It is convenient to define a *stretch* to be either an *open* interval or the limiting case of an open interval when one or more of the inequalities between the coordinates of a and b are replaced by equalities. Thus in one dimension a stretch would be either an open interval or a point; in two dimensions it would be an open interval, or an open side of an interval, or a point, etc.

When distinction is necessary, we use the symbol t to denote a point on the straight line.

Definition of a function. A quantity y is said to be a function of x, in an interval (a, b), if to each x of that interval corresponds a single value y.

We use the symbol $f(x)$ to denote a function of x.

A function is said to be *bounded* if the set of numbers consisting of all its values is bounded.

If corresponding to each point x we are given a succession of numbers $f_1(x), f_2(x), \ldots$, these may be said to define a *succession of functions*.

The upper bound $K(x)$ of the numbers corresponding to the point x define a new function, which we call the *upper bounding function* of the succession of functions.

Similarly we define the *lower bounding function*.

The upper limit $U(x)$ of the numbers $f_1(x), f_2(x), \ldots$ corresponding to the point x defines a new function, which we call the *upper function* of the succession of functions.

Similarly we define the *lower function*.

The succession of functions is called a *sequence*, if its upper and lower functions are identical. They are then called the *limiting function* of the sequence.

The succession of functions is said to be a *monotone ascending sequence* if the numbers corresponding to each x form an ascending sequence; it is said to be *monotone descending* if they form a monotone descending sequence. In either case the succession of functions is called a *monotone sequence of functions.*

Given any succession of functions, the upper bounding function $K_r(x)$ of the succession obtained by leaving out the r first functions, generates as r increases a monotone descending sequence which converges to the upper function $U(x)$, and the corresponding monotone ascending sequence of the lower bounding functions, converges to the lower function.

We have thus again reduced the consideration of all successions of functions to that of monotone sequences of functions.

In the case of monotone sequences of functions one bounding function is the first function, the other is the limiting function.

The convergence of a sequence of functions is said to be *bounded* if the bounding functions are bounded. The functions of the sequence are then said to be uniformly bounded.

The convergence of a sequence of functions $f_n(x)$ to a limiting function $f(x)$ in a closed interval is said to be *uniform* if, given any positive number e, an index N independent of x can be found such that, for all n from and after N,

$$|f_n(x) - f(x)| < e \text{ for all } x.$$

It is uniform in an open interval if uniform in every closed component interval. (See Appendix, 1, p. 44, last 18 lines and seq.)

§ 3. **Limits of a function at a point.** The upper and lower bounds of a set of numbers consisting of all the values of a function in an interval are called *the upper and lower bounds of the function in that interval.* Between them lie the bounds of the function in any interval interior to that interval.

Let x_0 be any point interior to the interval of definition (a, b). Let $w_1, w_2, \ldots, w_r, \ldots$ be any succession of intervals having x_0 as internal point and whose lengths converge to zero, each interval being contained in the preceding. Let M_r, m_r denote respectively the upper and lower bounds of the values of the function in w_r, *excluding the point* x_0. As r increases they form two monotone sequences of numbers and their limits, U and L respectively, are called the *upper* and the *lower limits* *of the function at the point* x_0. These limits are independent of the choice of the succession of intervals*.

For consider two successions of intervals $w_1, w_2, \ldots$ and $w_1', w_2', \ldots$; let the corresponding limits be U, L and U', L'.

There cannot be more than a finite number of intervals of the second succession not interior to w_r. Since r is arbitrary U', L' therefore lie between U and L.

But, reversing the rôles of the two successions, U, L lie between U', L'. Hence $U = U'$, $L = L'$.

If in the definition of upper and lower limits at a point x_0 instead of intervals containing the point x_0, we consider intervals (open or closed) having x_0 for a corner-point and as before each contained in the preceding and with length decreasing to zero—this defines corresponding to each quadrant† (open or closed) at x_0 a unique pair of *upper* and *lower limits* (or limits of approach) *in that* (open or closed) *quadrant*.

In the case of an internal point of the interval of definition the upper and lower limits at the point are respectively equal to the greatest and to the least of the upper and lower limits in the various *closed* quadrants at the point. We may use this to define the upper and lower limits at every point of the *closed* interval of definition.

We thus have, in all cases, corresponding to each point x of the interval of definition, three numbers, the value of the function and its upper and lower limits.

It is obvious that the greatest and the least‡ of these three numbers are respectively the limits of the upper and lower bounds of the function in any succession of intervals of lengths decreasing to zero, the point x being internal to all of them (the value at x is not excluded). If the

* Also each interval need not lie in the preceding provided they ultimately shrink up to x_0. For let u_1 be the smallest interval containing all those of the succession, u_2 the smallest interval containing all except w_1, etc. Then $u_1, u_2, \ldots$ are each inside the preceding and shrink up to x_0. Again if v_1 is w_1, v_2 the largest interval inside w_2 which has no points outside $w_1, \ldots, v_n$ the largest interval in w_n having no points outside v_{n-1}, then $v_1, v_2, \ldots$ are a succession of intervals each inside the pre ceding and shrinking up to x_0. Also clearly

$$u_n \geqslant w_n \geqslant v_n.$$

Hence the limits for w_n lie between the corresponding limits for u_n and v_n which coincide.

† By quadrant at a point we mean, in one dimension, one or other of the two sides of the point; in n dimensions, an angle determined by n parallels to the axes through the point.

‡ They are sometimes called the maximum and minimum limits and their difference is sometimes called the *jump* of the function at the point x.

value at the point is the greatest, it is also the limit of the upper bounds of the function in any succession of closed intervals which have x as corner-point, and whose lengths decrease to zero. In that case the function is said to be *upper semicontinuous* at the point x. Similarly, if the value of the function is the least of the three numbers, the function is said to be *lower semicontinuous* at the point.

Similarly we may define upper and lower semicontinuity in a closed or open quadrant at a point.

If a function is both upper and lower semicontinuous at a point and its value there is finite, it is said to be *continuous** at the point. Otherwise the function is *discontinuous* at the point.

§ 4. **Semicontinuity and the theorem of bounds.** A function is said to be upper semicontinuous in an interval if it is upper semicontinuous at every point of the interval. We shall call it a *U-function.*

A function is said to be lower semicontinuous in an interval if it is lower semicontinuous at every point of the interval. We shall call it an *L-function.*

In either case it is said to be *semicontinuous* in the interval. A function which is both an L and a U and assumes only finite values is said to be *continuous.*

e-definition of semicontinuity at a point. A function $f(x)$ is said to be upper semicontinuous at the point x_0 if, given any positive quantity e, there is an interval having x_0 as middle point throughout which if $f(x_0)$ is finite

while, if $f(x_0)$ is $-\infty$,

$$\left.\begin{aligned} f(x) &\leqslant f(x_0)+e, \\ f(x) &\leqslant -1/e \end{aligned}\right\} (U).$$

A function $f(x)$ is said to be lower semicontinuous at the point x_0 if, given any positive quantity e, there is an interval having x_0 as middle point, throughout which, if $f(x_0)$ is finite,

while, if $f(x_0)$ is $+\infty$,

$$\left.\begin{aligned} f(x) &\geqslant f(x_0)-e, \\ f(x) &\geqslant 1/e \end{aligned}\right\} (L).$$

If $f(x_0)$ is $+\infty$, then $f(x)$ is certainly upper semicontinuous at x_0; if $f(x_0)$ is $-\infty$, then $f(x)$ is certainly lower semicontinuous at x_0.

THEOREM. An L-function assumes its lower bound in every closed interval; a U-function, its upper bound.

Divide the given interval into two equal parts. If m is the lower bound of our L-function in the given closed interval, it is also its lower

* See Appendix, 1.

bound in at least one of these closed subintervals. Let W_1 be the first having this property. Again divide W_1 into two equal parts, and let W_2 be the first of these closed subintervals of W_1 in which m is again the lower bound. And so on.

The succession of intervals

$$W_1,\ W_2,\ \ldots,\ W_n,\ \ldots$$

consists of closed intervals, each contained in the preceding, and whose lengths decrease to zero. There is exactly one point common to all of them. The value of our L-function at that point is therefore the limit of its lower bounds in the succession of intervals, that is to say m. Q.E.D.

Similarly, we may prove the corresponding theorem for U-functions.

The theorem of bounds.

If $f_1(x)$, $f_2(x)$, ..., is a monotone ascending sequence of functions having $f(x)$ as limiting function; if u_n and l_n are respectively the upper and the lower bounds of $f_n(x)$ in a fixed closed interval, u and l those of $f(x)$ in the same interval, then

$$\lim u_n = u \quad \text{while} \quad \lim l_n \leqslant l.$$

Moreover, if the functions are all L-functions, then

$$\lim l_n = l.$$

It is obvious that if f_1 is never greater than f_2, the same is true of their bounds. Hence

$$u_1 \leqslant u_2 \leqslant \ldots \leqslant u; \quad l_1 \leqslant l_2 \leqslant \ldots \leqslant l.$$

Therefore $$\lim u_n \leqslant u; \quad \lim l_n \leqslant l.$$

But if A is any quantity less than u, there are points x such that $f(x) > A$. At any such point x, we have, since $f_n(x)$ converges to $f(x)$, $f_n(x) > A$ from and after a certain $f_N(x)$. Therefore, a fortiori,

$$\lim u_n > A,$$

or, since A was any quantity less than u, $\lim u_n \geqslant u$.

Therefore $$\lim u_n = u, \quad \text{while} \quad \lim l_n \leqslant l.$$

In the case in which the functions f_n are L-functions, we can find a point x_n where f_n assumes its lower bound. Let x' be any limiting point of the x_n, and let B be any quantity less than $f(x')$. Then

$$f_n(x') > B$$

from and after a certain $f_{N'}$. There is therefore an interval surrounding x' throughout which, since $f_{N'}$ is an L-function, $f_{N'}(x)$ is greater than B. In that same interval, by monotony,

$$f_n(x) > B$$

from and after $f_{N'}$. In this interval there are an infinite number of the points x_n with indices greater than N'. Therefore, if N'' is the first of these, then

$$f_{N''}(x_{N''}) = l_{N''} > B.$$

Therefore, a fortiori,

$$\lim l_n \geqslant B.$$

Since B was any number less than $f(x')$,

$$f(x') \leqslant \lim l_n,$$

and, a fortiori,

$$l \leqslant \lim l_n.$$

CHAPTER II

THE GENERATION OF FUNCTIONS

§ 1. **The simple functions.** The simplest functions are the constants; the value of y is the same for all x.

The next simplest functions are *the functions constant in stretches,* whose interval of definition is the sum of a finite number of stretches inside each of which the function is a finite constant. By a stretch we mean, as explained on p. 4, an open interval or a kind of limiting case of an open interval, such as a point.

A function constant in stretches is not in general semicontinuous.

For example, the function defined in the interval (0, 1), whose value in the open interval (0, 1/2) is 0, whose value at the point 1/2 is 1/2, whose value in the open interval (1/2, 1) is 1, is not semicontinuous at the point 1/2.

A function constant in stretches will certainly be lower semicontinuous at every point of its interval of definition if its value in every stretch which is not an interval is equal to the least of the values in the neighbouring intervals. It is then called a *simple L-function.*

Similarly, we call *simple U-function* a function constant in stretches whose value in every stretch which is not an interval is the greatest of the values in the neighbouring intervals.

These two types of *simple functions* have the following properties:

(i) The sum of two functions of the same type is of that type.

(ii) The function equal to the greater of two functions of the same type, and the function equal to the smaller of two functions of the same type, at each point, are of that type.

(iii) Change of sign transfers each type to the twin type.

These properties we shall refer to as *the three fundamental properties of class.*

§ 2. The generation of general semicontinuous functions by monotone sequences of simple functions.

THEOREM. Any L-function bounded below is expressible as the limit of a monotone ascending sequence of simple L-functions and also as the limit of such a sequence of simple U-functions.

Let $f(x)$ be the given L-function, defined in the interval (a, b). Let m be its lower bound.

We divide (a, b) into two equal parts and we call m'_1, m''_1 the lower bounds of $f(x)$ in each of these parts, the common boundary points being taken to belong to both.

We again bisect each of the parts.

(In the case of several variables we bisect in turn the range of each of them.)

Let m'_2, m''_2, m'''_2, m''''_2, be the bounds so obtained, and so on.

If at any stage, the nth say, one of these numbers be infinite, we replace it by the greatest of all the preceding plus n. In that case $f(x)$ would have to be infinite ($+\infty$) in the whole of the corresponding interval.

Let $a_0(x)$, $b_0(x)$ be the constant m.

Let $a_1(x)$, $b_1(x)$ be respectively the simple L-function and the simple U-function which are equal to m'_1 in the first half of (a, b) and to m''_1 in the second half, their values at the common boundary points* being of course respectively the smallest and the largest of these two numbers.

Let $a_2(x)$, $b_2(x)$ be respectively the simple L-function and the simple U-function equal to m'_2 in the first quarter of (a, b), to m''_2 in the second, and so on.

The two sequences of functions,

$$a_0(x),\quad a_1(x),\quad a_2(x),\ \ldots,$$
$$b_0(x),\quad b_1(x),\quad b_2(x),\ \ldots,$$

are monotone ascending. The values of their limiting function at any point x are in both cases equal to the limits of the lower bounds of $f(x)$ in one or more successions of closed intervals each contained in the preceding and whose lengths tend to zero, and such that x is common to all of them.

Since $f(x)$ is semicontinuous the limiting functions therefore both coincide with $f(x)$. Q.E.D.

Similarly we can establish the corresponding

* At which alone they differ.

THEOREM. Any U-function bounded above is expressible as the limit of a monotone descending sequence of simple U-functions, and also as the limit of such a sequence of simple L-functions.

The converse of these theorems is not true. Monotone sequences of simple functions do not in general define semicontinuous functions. It is, however, easy to show that monotone ascending sequences of simple L-functions always define L-functions, while monotone descending sequences of simple U-functions always define U-functions. More generally, we have

THEOREM. The limiting function of a monotone ascending sequence of L-functions is an L-function.

Let $f_n(x)$ be the generic function of the sequence, and $f(x)$ the limiting function. Since f is never less than f_n, the same is true of their lower bounds in any interval and consequently of their lower limits at any point. Let l be the lower limit of $f(x_0)$ at the point x_0. Since f_n is semicontinuous, $f_n(x_0)$ is smaller than or equal to the lower limit of f_n at x_0. Therefore, a fortiori,

$$f_n(x_0) \leqslant l.$$

This is true for all n. Therefore

$$f(x_0) \leqslant l.$$

Similarly we can prove the corresponding

THEOREM. The limit of a monotone descending sequence of U-functions is itself a U-function.

The two types of semicontinuous functions are easily seen to possess the *three fundamental properties of class.*

i. The sum of two functions of the same type is of that type.

If f and g are two L-functions bounded below they are the limits of two monotone sequences of simple L-functions

$$f_1, f_2, \ldots, \quad g_1, g_2, \ldots,$$

their sum is therefore the limit of the monotone ascending sequence

$$f_1+g_1, \quad f_2+g_2, \ldots$$

of simple L-functions, i.e. an L-function*.

If f and g are not bounded below, every point, where neither assumes the value $-\infty$, is internal to an interval where both are bounded below.

At a point where one assumes this value, the other being different from $+\infty$, the sum assumes the value $-\infty$, and is consequently lower semicontinuous.

At a point where one function assumes the value $-\infty$, and the other the value $+\infty$, the sum is not defined.

Similarly we prove the theorem for U-functions.

* The same reasoning establishes the property for any type of functions defined by monotone sequences of functions belonging to a type which has the property.