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Lars Hörmander

The Analysis of Linear Partial Differential Operators I

线性偏微分算子分析

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The Analysis of Linear Partial Differential Operators I

Distribution Theory and Fourier Analysis

Second Edition

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Preface to the Second Edition

The main change in this edition is the inclusion of exercises with answers and hints. This is meant to emphasize that this volume has been written as a general course in modern analysis on a graduate student level and not only as the beginning of a specialized course in partial differential equations. In particular, it could also serve as an introduction to harmonic analysis. Exercises are given primarily to the sections of general interest; there are none to the last two chapters.

Most of the exercises are just routine problems meant to give some familiarity with standard use of the tools introduced in the text. Others are extensions of the theory presented there. As a rule rather complete though brief solutions are then given in the answers and hints.

To a large extent the exercises have been taken over from courses or examinations given by Anders Melin or myself at the University of Lund. I am grateful to Anders Melin for letting me use the problems originating from him and for numerous valuable comments on this collection.

As in the revised printing of Volume II, a number of minor flaws have also been corrected in this edition. Many of these have been called to my attention by the Russian translators of the first edition, and I wish to thank them for our excellent collaboration.

Lund, October 1989 Lars Hörmander

Preface

In 1963 my book entitled "Linear partial differential operators" was published in the Grundlehren series. Some parts of it have aged well but others have been made obsolete for quite some time by techniques using pseudo-differential and Fourier integral operators. The rapid development has made it difficult to bring the book up to date. However, the new methods seem to have matured enough now to make an attempt worth while.

The progress in the theory of linear partial differential equations during the past 30 years owes much to the theory of distributions created by Laurent Schwartz at the end of the 1940's. It summed up a great deal of the experience accumulated in the study of partial differential equations up to that time, and it has provided an ideal framework for later developments. "Linear partial differential operators" began with a brief summary of distribution theory for this was still unfamiliar to many analysts 20 years ago. The presentation then proceeded directly to the most general results available on partial differential operators. Thus the reader was expected to have some prior familiarity with the classical theory although it was not appealed to explicitly. Today it may no longer be necessary to include basic distribution theory but it does not seem reasonable to assume a classical background in the theory of partial differential equations since modern treatments are rare. Now the techniques developed in the study of singularities of solutions of differential equations make it possible to regard a fair amount of this material as consequences of extensions of distribution theory. Rather than omitting distribution theory I have therefore decided to make the first volume of this book a greatly expanded treatment of it. The title has been modified so that it indicates the general analytical contents of this volume. Special emphasis is put on Fourier analysis, particularly results related to the stationary phase method and Fourier analysis of singularities. The theory is illustrated throughout with examples taken from the theory of partial differential equations. These scattered examples should give a sufficient knowledge of the classical theory to serve as an introduction to the systematic study in the later volumes. Volume I should also be a useful introduction to harmonic analysis. A chapter on hyperfunctions at the end is intended to give an introduction in the spirit of Schwartz distributions to this subject and to the analytic theory of partial differential equations. The great progress in this area due primarily to the school of Sato is beyond the scope of this book, however.

The second and the third volumes will be devoted to the theory of differential equations with constant and with variable coefficients respectively. Their prefaces will describe their contents in greater detail. Volume II will appear almost simultaneously with Volume I, and Volume III will hopefully be published not much more than two years later.

In a work of this kind it is not easy to provide adequate references. Many ideas and methods have evolved slowly for centuries, and it is a task for a historian of mathematics to uncover the development completely. Also the more recent history provides of course considerable difficulties in establishing priorities correctly, and these problems tend to be emotionally charged. All this makes it tempting to omit references altogether. However, rather than doing so I have chosen to give at the end of each chapter a number of references indicating recent sources for the material presented or closely related topics. Some references to the earlier literature are also given. I hope this will be helpful to the reader interested in examining the background of the results presented, and I also hope to be informed when my references are found quite inadequate so that they can be improved in a later edition.

Many colleagues and students have helped to improve this book, and I should like to thank them all. The discussion of the analytic wave front sets owes much to remarks by Louis Boutet de Monvel, Pierre Schapira and Johannes Sjöstrand. A large part of the manuscript was read and commented on by Anders Melin and Ragnar Sigurdsson in Lund, and Professor Wang Rou-hwai of Jilin University has read a large part of the proofs. The detailed and constructive criticism given by the participants in a seminar on the book conducted by Gerd Grubb at the University of Copenhagen has been a very great help. Niels Jørgen Kokholm took very active part in the seminar and has also read all the proofs. In doing so he has found a number of mistakes and suggested many improvements. His help has been invaluable to me.

Finally, I wish to express my gratitude to the Springer Verlag for encouraging me over a period of years to undertake this project and for first rate and patient technical help in its execution.

Lund, January 1983 Lars Hörmander

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Contents

Introduction

In differential calculus one encounters immediately the unpleasant fact that not every function is differentiable. The purpose of distribution theory is to remedy this flaw; indeed, the space of distributions is essentially the smallest extension of the space of continuous functions where differentiation is always well defined. Perhaps it is therefore self evident that it is desirable to make such an extension, but let us anyway discuss some examples of how awkward it is not to be allowed to differentiate.

Our first example is the Fourier transformation which will be studied in Chapter VII. If v is an integrable function on the real line then the Fourier transform Fv is the continuous function defined by

$$(Fv)(\xi)=\int_{-\infty}^{\infty} e^{-ix\xi}v(x)\,dx, \qquad \xi\in\mathbb{R}.$$

It has the important property that

$$F(Dv)=MFv, \qquad F(Mv)=-DFv \tag{1}$$

whenever both sides are defined; here $Dv(x)=-i\,dv/dx$ and $Mv(x)=xv(x)$. In the first formula the multiplication operator M is always well defined so the same ought to be true for D. Incidentally the second formula (1) then suggests that one should also define F for functions of polynomial increase.

Next we shall examine some examples from the theory of partial differential equations which also show the need for a more general definition of derivatives. Classical solutions of the Laplace equation

$$\partial^2 u/\partial x^2+\partial^2 u/\partial y^2=0 \tag{2}$$

or the wave equation (in two variables)

$$\partial^2 v/\partial x^2-\partial^2 v/\partial y^2=0 \tag{3}$$

are twice continuously differentiable functions satisfying the equations everywhere. It is easily shown that uniform limits of classical solutions of the Laplace equation are classical solutions. On the other hand, the classical solutions of the wave equation are all functions of the form

$$v(x, y) = f(x+y) + g(x-y) \tag{4}$$

with twice continuously differentiable f and g, and they have as uniform limits all functions of the form (4) with f and g continuous. All such functions ought therefore to be recognized as solutions of (3) so the definition of a classical solution is too restrictive.

Let us now consider the corresponding inhomogeneous equations

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = F, \tag{5}$$

$$\partial^2 v/\partial x^2 - \partial^2 v/\partial y^2 = F \tag{6}$$

where F is a continuous function vanishing outside a bounded set. If F is continuously differentiable a solution of (6) is given by

$$v(x, y) = \iint_{\eta - y + |x-\xi| < 0} -F(\xi, \eta)\, d\xi\, d\eta/2. \tag{7}$$

However, (7) defines a continuously differentiable function v even if F is just continuous. Clearly we must accept v as a solution of (6) even if second order derivatives do not exist. Similarly (5) has the classical solution

$$u(x, y) = (4\pi)^{-1} \iint F(\xi, \eta) \log((x-\xi)^2 + (y-\eta)^2)\, d\xi\, d\eta \tag{8}$$

provided that F is continuously differentiable. Again (8) defines a continuously differentiable function u even if F is just continuous, and we should be able to accept u as a solution of (5) also in that case.

The difficulties which are illustrated in their simplest form by the preceding examples were eliminated already by the concept of *weak solution* which preceded distribution theory. The idea is to rewrite the equation considered in a form where the unknown function u is no longer differentiated. Consider as an example the equation (6). If u is a classical solution it follows that

$$\iint (\partial^2 u/\partial x^2 - \partial^2 u/\partial y^2)\, \phi\, dx\, dy = \iint F \phi\, dx\, dy \tag{6$'$}$$

for every continuous function ϕ vanishing outside a compact, that is, closed and bounded, set. Conversely, if (6)′ is fulfilled for all such ϕ which are say twice continuously differentiable then (6) is fulfilled. In fact, if (6) were not satisfied at a point (x_0, y_0) we could take ϕ non-negative and 0 outside a small neighborhood of (x_0, y_0) and conclude that (6)′ is not fulfilled either. For such "test functions" ϕ we can integrate by parts twice in the left-hand side of (6)′ which gives the equivalent formula

(6)'' $$\iint u(\partial^2\phi/\partial x^2-\partial^2\phi/\partial y^2)\,dx\,dy=\iint F\phi\,dx\,dy.$$

Summing up, if u is twice continuously differentiable then (6) is equivalent to the validity of (6)'' for all test functions ϕ, that is, twice continuously differentiable functions ϕ vanishing outside a compact set. However, (6)'' has a meaning if u is just continuous, and one calls u a weak solution of (6) when (6)'' is valid.[1] It is easily verified that the flaws of the classical solutions pointed out above disappear if one accepts weak continuous solutions.

The function F is uniquely determined by u when (6)'' is fulfilled. However, for an arbitrary continuous function u there may be no continuous function F such that

(9) $$L(\phi)=\iint u(\partial^2\phi/\partial x^2-\partial^2\phi/\partial y^2)\,dx\,dy$$

can be written in the form

(10) $$L(\phi)=\iint F\phi\,dx\,dy.$$

Distribution theory goes beyond the definition of weak solutions by accepting for study expressions L of the form (9) even when they are not of the form (10). A distribution is any such expression which depends linearly on a test function ϕ (and its derivatives). When it can be written in the form (10) it is identified with the function F. It turns out that one can extend the basic operations of calculus to distributions; in particular differentiation is always defined.

Let us also consider some examples of similar expressions occurring in physics. First consider a point mass with weight 1 at a point a on the real axis. This can be considered as a limiting case of a mass distribution with uniform density $1/2\varepsilon$ in the interval $(a-\varepsilon,\ a+\varepsilon)$ as $\varepsilon\to 0$. The corresponding functional is

$$L_\varepsilon(\phi)=\int_{a-\varepsilon}^{a+\varepsilon}\phi(x)\,dx/2\varepsilon.$$

When $\varepsilon\to 0$ we have $L_\varepsilon(\phi)\to\phi(a)$, so $L(\phi)=\phi(a)$ should represent the unit mass at a. This interpretation is of course standard in measure theory.

Next we consider a dipole at 0 with moment 1. This may be defined as the limit of the pointmass $1/\delta$ at δ and $-1/\delta$ at 0 as $\delta\to 0$. Thus we must consider the limit of the functional

$$M_\delta(\phi)=\delta^{-1}\phi(\delta)-\delta^{-1}\phi(0)$$

[1] Note that differential equations appear naturally in a weak form in the calculus of variations.

when $\delta \to 0$, which is $M(\phi)=\phi'(0)$. This functional is therefore the appropriate description of the dipole.

It is possible to pursue this development and define distributions as limits of functionals of the form (10). However, we shall not do so but rather follow the path suggested by the definition of weak derivatives. This is the original definition of Schwartz and it has the advantage of avoiding the question which limits define the same distribution.

The formal definition of distributions is given in Chapter II after properties of test functions have been discussed at some length in Chapter I. Differentiation of distributions is then studied in Chapter III; it is shown in Section 4.4 that we have indeed obtained a minimal extension of the space of continuous functions where differentiation is always possible. In Chapters IV, V, VI we extend convolution, direct product and composition from functions to distributions. Chapter VII is devoted to Fourier analysis of functions and distributions. The choice of material differs a great deal from standard texts since it is dictated by what is required in the later parts. The method of stationary phase is given a particularly thorough treatment. In Chapter VIII we discuss the Fourier analysis of singularities of distributions. This turns out to be a local problem so it can be discussed also for distributions on manifolds. The phrase "singularity" above is deliberately vague; in fact we shall consider singularities both from a C^∞ and from an analytic point of view. The results lead to important extensions of the distribution theory in Chapters III–VI. For instance, one can define the restriction of a distribution u to a submanifold Y if u has no singularity at a normal to Y. Many applications to regularity and uniqueness of solutions of differential equations are also given. The analytic theory is continued in Chapter IX which is devoted to hyperfunctions. These are defined just as distributions but with real analytic test functions. The main new difficulty stems from the fact that there are no such test functions vanishing outside a compact set.

Chapter I. Test Functions

Summary

As indicated in the introduction one must work consistently with smooth "test functions" in the theory of distributions. In this chapter we have collected the basic facts that one needs to know about such functions. As an introduction a brief summary of differential calculus is given in Section 1.1. It is written with a reader in mind who has seen the material before but perhaps with different emphasis and different notation. The reader who finds the presentation hard to follow is recommended to study first a more thorough modern treatment of differential calculus in several variables, and experienced readers should proceed directly to Section 1.2. In addition to the basic indispensible facts we have included in Sections 1.3 and 1.4 some more refined constructions which will be useful some time in this book but are not important for the main theme. The reader in a hurry may therefore wish to omit Section 1.3 from Theorem 1.3.5 on and also Theorem 1.4.2, Lemma 1.4.3 and the rest of Section 1.4 from Theorem 1.4.6 on.

1.1. A Review of Differential Calculus

At first we shall consider functions of a single real variable but permit values in a Banach space. Thus let I be an open interval on the real line $\mathbb{R}$ and let V be a Banach space with norm denoted $\| \;\|$. A map f: $I \to V$ is called differentiable at $x \in I$, with derivative $f'(x) \in V$, if

$$\|(f(x+h)-f(x))/h - f'(x)\| \to 0 \quad \text{when } h \to 0. \tag{1.1.1}$$

We can write (1.1.1) in the equivalent form

$$\|f(x+h)-f(x)-f'(x)\,h\| = o(|h|) \quad \text{when } h \to 0. \tag{1.1.1$'$}$$

If $V=\mathbb{R}^n$ and we write $f=(f_1, \dots, f_n)$ then differentiability of f is of course equivalent to differentiability of each component f_j. For vector valued functions the mean value theorem must be replaced by the following

Theorem 1.1.1. *If $f: I \to V$ is differentiable at every point in I, then*

$$\|f(y)-f(x)\| \leqq |y-x| \sup\{\|f'(x+t(y-x))\|, 0 \leqq t \leqq 1\}; \quad x, y \in I. \tag{1.1.2}$$

Proof. Let $M > \sup\{\|f'(x+t(y-x))\|, 0 \leqq t \leqq 1\}$ and set

$$E = \{t; 0 \leqq t \leqq 1, \|f(x+t(y-x))-f(x)\| \leqq Mt|x-y|\}.$$

E is closed since f is continuous, and $0 \in E$, so E has a largest element s. If $t > s$ and $t-s$ is sufficiently small we have

$$\begin{aligned} &\|f(x+t(y-x))-f(x)\| \\ &\quad \leqq \|f(x+t(y-x))-f(x+s(y-x))\| + \|f(x+s(y-x))-f(x)\| \\ &\quad \leqq M|(t-s)(y-x)| + Ms|y-x| = Mt|y-x|. \end{aligned}$$

Hence $s=1$ which proves the theorem.

Remark. If f is just continuous in $[x, y]$ and differentiable in the interior we obtain (1.1.2) with supremum for $0<t<1$ as a limit of (1.1.2) applied to smaller intervals. If $v \in V$ an application of (1.1.2) to $x \to f(x) - xv$ gives

$$\|f(y)-f(x)-v(y-x)\| \leqq |y-x| \sup_{0<t<1} \|f'(x+t(y-x))-v\| \tag{1.1.2$'$}$$

which is often more useful, particularly with $v = f'(x)$.

Corollary 1.1.2. *Let f be continuous in I and differentiable outside a closed subset F where $f=0$. If $x \in F$ and $f'(y) \to 0$ when $I \setminus F \ni y \to x$, then $f'(x)$ exists and is equal to 0.*

Proof. If $y \in F$ then $f(y)-f(x)=0$. Otherwise let z be the point in $F \cap [x, y]$ closest to y. Then (1.1.2)$'$ gives

$$\|f(y)-f(x)\| = \|f(y)-f(z)\| \leqq |y-z| \sup_{0<t<1} \|f'(z+t(y-z))\|$$

which is $o(|y-x|)$ as $y \to x$.

Example 1.1.3. If P is a polynomial and $f(x) = P(1/x)e^{-1/x}$, $x>0$, $f(x) = 0$, $x \leqq 0$, then f is continuous. The derivative for $x \neq 0$ is of the same form with $P(1/x)$ replaced by $(P(1/x) - P'(1/x))/x^2$ so $f'(0)$ exists and is equal to 0.