Groups, Languages and Automata

DEREK F. HOLT, SARAH REES
and CLAAS E. RÖVER

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Fascinating connections exist between group theory and automata theory, and a wide variety of them are discussed in this text. Automata can be used in group theory to encode complexity, to represent aspects of underlying geometry on a space on which a group acts, and to provide efficient algorithms for practical computation. There are also many applications in geometric group theory.

The authors provide background material in each of these related areas, as well as exploring the connections along a number of strands that lead to the forefront of current research in geometric group theory. Examples studied in detail include hyperbolic groups, Euclidean groups, braid groups, Coxeter groups, Artin groups, and automata groups such as the Grigorchuk group. This book will be a convenient reference point for established mathematicians who need to understand background material for applications, and can serve as a textbook for research students in (geometric) group theory.

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Preface

This book explores connections between group theory and automata theory. We were motivated to write it by our observations of a great diversity of such connections; we see automata used to encode complexity, to recognise aspects of underlying geometry, to provide efficient algorithms for practical computation, and more.

The book is pitched at beginning graduate students, and at professional academic mathematicians who are not familiar with all aspects of these interconnected fields. It provides background in automata theory sufficient for its applications to group theory, and then gives up-to-date accounts of these various applications. We assume that the reader already has a basic knowledge of group theory, as provided in a standard undergraduate course, but we do not assume any previous knowledge of automata theory.

The groups that we consider are all finitely generated. An element of a group G is represented as a product of powers of elements of the generating set X, and hence as a string of symbols from $A := X \cup X^{-1}$, also called words. Many different strings may represent the same element. The group may be defined by a presentation; that is, by its generating set X together with a set R of relations, from which all equations in the group between strings can be derived. Alternatively, as for instance in the case of automata groups, G might be defined as a group of functions generated by the elements of X.

Certain sets of strings, also called languages, over A are naturally of interest. We study the word problem of the group G, namely the set $\mathrm{WP}(G, A)$ of strings over A that represent the identity element. We define a language for G to be a language over A that maps onto G, and consider the language of all geodesics, and various languages that map bijectively to G. We also consider combings, defined to be group languages for which two words representing either the same element or elements that are adjacent in the Cayley graph fellow travel; that is, they are at a bounded distant apart throughout their length.

We consider an automaton to be a device for defining a, typically infinite, set L of strings over a finite alphabet, called the language of the automaton. Any string over the finite alphabet may be input to the automaton, and is then either accepted if it is in L, or rejected if it is not. We consider automata of varying degrees of complexity, ranging from finite state automata, which define regular languages, through pushdown automata, defining context-free languages, to Turing machines, which set the boundaries for algorithmic recognition of a language. In other words, we consider the full Chomsky hierarchy of formal languages and the associated models of computation.

Finite state automata were used by Thurston in his definition of automatic groups after he realised that both the fellow traveller property and the finiteness of the set of cone types that Cannon had identified in the fundamental groups of compact hyperbolic manifolds could be expressed in terms of regular languages. For automatic groups a regular combing can be found; use of the finite state automaton that defines this together with other automata that encode fellow travelling allows in particular a quadratic time solution to the word problem. Word-hyperbolic groups, as defined by Gromov, can be characterised by their possession of automatic structures of a particular type, leading to linear time solutions to the word problem.

For some groups the set of all geodesic words over A is a regular language. This is true for word-hyperbolic groups and abelian groups, with respect to any generating set, and for many other groups, including Coxeter groups, virtually free groups and Garside groups, for certain generating sets. Many of these groups are in fact automatic.

The position of a language in the Chomsky hierarchy can be used as a measure of its complexity. For example, the problem of deciding whether an input word w over A represents the identity of G (which, like the set it recognises, is called the word problem) can be solved by a terminating algorithm if and only if the set $\mathrm{WP}(G,A)$ and its complement can be recognised by a Turing machine; that is, if and only if $\mathrm{WP}(G,A)$ is recursive. We also present a proof of the well-known fact that finitely presented groups exist for which the word problem is not soluble; the proof of this result encodes the existence of Turing machines with non-recursive languages.

When the word problem is soluble, some connections can be made between the position of the language $\mathrm{WP}(G,A)$ in the Chomsky hierarchy and the algebraic properties of the group. It is elementary to see that $\mathrm{WP}(G,A)$ is regular if and only if G is finite, while a highly non-trivial result of Muller and Schupp shows that a group has context-free word problem if and only if it has a free subgroup of finite index.

Attaching an output-tape to an automaton extends it from a device that defines a set of strings to a function from one set of strings to another. We call such a device a transducer, and show how transducers can be used to define groups. Among these groups are finitely generated infinite torsion groups, groups of intermediate growth, groups of non-uniform exponential growth, iterated monodromy groups of post-critically finite self-coverings of the Riemann sphere, counterexamples to the strong Atiyah conjecture, and many others. Our account is by no means complete, as it concentrates on introducing terminology, the exposition of some basic techniques and pointers to the literature.

There is a shorter book by Ian Chiswell [68] that covers some of the same material as we do, including groups with context-free word problem and an introduction to the theory of automatic groups. Our emphasis is on connections between group theory and formal language theory rather than computational complexity, but there is a significant overlap between these areas. We recommend also the article by Mark Sapir [226] for a survey of results concerning the time and space complexity of the fundamental decision problems in group theory.

Acknowledgements We are grateful to Professor Rick Thomas for numerous helpful and detailed discussions on the topics covered in this book. Some of the text in Section 1.2, and the statement and proofs of Propositions 3.4.9 and 3.4.10 were written originally by Rick.

We are grateful also to Professor Susan Hermiller and to an anonymous reviewer for a variety of helpful comments.

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PART ONE

INTRODUCTION