

Groups, Languages and Automata

DEREK F. HOLT, SARAH REES
and CLAAS E. RÖVER

London Mathematical Society
Student Texts **88**



LONDON
MATHEMATICAL
SOCIETY
EST. 1865

CAMBRIDGE

Fascinating connections exist between group theory and automata theory, and a wide variety of them are discussed in this text. Automata can be used in group theory to encode complexity, to represent aspects of underlying geometry on a space on which a group acts, and to provide efficient algorithms for practical computation. There are also many applications in geometric group theory.

The authors provide background material in each of these related areas, as well as exploring the connections along a number of strands that lead to the forefront of current research in geometric group theory. Examples studied in detail include hyperbolic groups, Euclidean groups, braid groups, Coxeter groups, Artin groups, and automata groups such as the Grigorchuk group. This book will be a convenient reference point for established mathematicians who need to understand background material for applications, and can serve as a textbook for research students in (geometric) group theory.

London Mathematical Society Student Texts

Edited by Professor D. J. BENSON,
Department of Mathematics,
University of Aberdeen, Meston Building, King's College
Aberdeen AB24 3UE, United Kingdom

with assistance from

A. M. Etheridge (<i>Oxford</i>)	M. W. Liebeck (<i>Imperial</i>)
J. R. Partington (<i>Leeds</i>)	R. M. Roberts (<i>Surrey</i>)
U. L. Tillmann (<i>Oxford</i>)	J. F. Toland (<i>Bath</i>)

This series consists of textbooks aimed at advanced undergraduates or beginning graduate students. It covers the whole range of pure mathematics, as well as topics in applied mathematics and mathematical physics that involve a substantial use of modern mathematical methods.

Topics not of a standard nature, though of current interest, are also covered by volumes in this series.

CAMBRIDGE
UNIVERSITY PRESS
www.cambridge.org

ISBN 978-1-316-60652-0



9 781316 606520 >

LMSST 88 HOLT, REES and RÖVER Groups, Languages and Automata CAMBRIDGE

London Mathematical Society Student Texts 88

Groups, Languages and Automata

DEREK F. HOLT

University of Warwick

SARAH REES

University of Newcastle upon Tyne

CLAAS E. RÖVER

National University of Ireland, Galway



CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom
One Liberty Plaza, 20th Floor, New York, NY 10006, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
4843/24, 2nd Floor, Ansari Road, Daryaganj, Delhi - 110002, India
79 Anson Road, #06-04/06, Singapore 079906

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning and research at the highest international levels of excellence.

www.cambridge.org

Information on this title: www.cambridge.org/9781107152359

DOI: 10.1017/9781316588246

© Derek F. Holt, Sarah Rees and Claas E. Röver 2017

This publication is in copyright. Subject to statutory exception and to the provisions of relevant collective licensing agreements, no reproduction of any part may take place without the written permission of Cambridge University Press.

First published 2017

Printed in the United Kingdom by Clays, St Ives plc

A catalogue record for this publication is available from the British Library.

ISBN 978-1-107-15235-9 Hardback

ISBN 978-1-316-60652-0 Paperback

Cambridge University Press has no responsibility for the persistence or accuracy of URLs for external or third-party Internet Web sites referred to in this publication and does not guarantee that any content on such Web sites is, or will remain, accurate or appropriate.

LONDON MATHEMATICAL SOCIETY STUDENT TEXTS

Managing Editor: Professor D. Benson,
Department of Mathematics, University of Aberdeen, UK

- 49 An introduction to K-theory for C^* -algebras, M. RØRDAM, F. LARSEN & N. J. LAUSTSEN
- 50 A brief guide to algebraic number theory, H. P. F. SWINNERTON-DYER
- 51 Steps in commutative algebra: Second edition, R. Y. SHARP
- 52 Finite Markov chains and algorithmic applications, OLLE HÄGGSTRÖM
- 53 The prime number theorem, G. J. O. JAMESON
- 54 Topics in graph automorphisms and reconstruction, JOSEF LAURI & RAFFAELE SCAPELLATO
- 55 Elementary number theory, group theory and Ramanujan graphs, GIULIANA DAVIDOFF, PETER SARNAK & ALAIN VALETTE
- 56 Logic, induction and sets, THOMAS FORSTER
- 57 Introduction to Banach algebras, operators and harmonic analysis, GARTH DALES *et al*
- 58 Computational algebraic geometry, HAL SCHENCK
- 59 Frobenius algebras and 2-D topological quantum field theories, JOACHIM KOCK
- 60 Linear operators and linear systems, JONATHAN R. PARTINGTON
- 61 An introduction to noncommutative Noetherian rings: Second edition, K. R. GOODEARL & R. B. WARFIELD, JR
- 62 Topics from one-dimensional dynamics, KAREN M. BRUCKS & HENK BRUIN
- 63 Singular points of plane curves, C. T. C. WALL
- 64 A short course on Banach space theory, N. L. CAROTHERS
- 65 Elements of the representation theory of associative algebras I, IBRAHIM ASSEM, DANIEL SIMSON & ANDRZEJ SKOWROŃSKI
- 66 An introduction to sieve methods and their applications, ALINA CARMEN COJOCARU & M. RAM MURTY
- 67 Elliptic functions, J. V. ARMITAGE & W. F. EBERLEIN
- 68 Hyperbolic geometry from a local viewpoint, LINDA KEEN & NIKOLA LAKIC
- 69 Lectures on Kähler geometry, ANDREI MOROIANU
- 70 Dependence logic, JOUKU VÄÄNÄNEN
- 71 Elements of the representation theory of associative algebras II, DANIEL SIMSON & ANDRZEJ SKOWROŃSKI
- 72 Elements of the representation theory of associative algebras III, DANIEL SIMSON & ANDRZEJ SKOWROŃSKI
- 73 Groups, graphs and trees, JOHN MEIER
- 74 Representation theorems in Hardy spaces, JAVAD MASHREGHI
- 75 An introduction to the theory of graph spectra, DRAGOŠ CVETKOVIĆ, PETER ROWLINSON & SLOBODAN SIMIĆ
- 76 Number theory in the spirit of Liouville, KENNETH S. WILLIAMS
- 77 Lectures on profinite topics in group theory, BENJAMIN KLOPSCH, NIKOLAY NIKOLOV & CHRISTOPHER VOLL
- 78 Clifford algebras: An introduction, D. J. H. GARLING
- 79 Introduction to compact Riemann surfaces and dessins d'enfants, ERNESTO GIRONDO & GABINO GONZÁLEZ-DIEZ
- 80 The Riemann hypothesis for function fields, MACHIEL VAN FRANKENHUIJSEN
- 81 Number theory, Fourier analysis and geometric discrepancy, GIANCARLO TRAVAGLINI
- 82 Finite geometry and combinatorial applications, SIMEON BALL
- 83 The geometry of celestial mechanics, HANSJÖRG GEIGES
- 84 Random graphs, geometry and asymptotic structure, MICHAEL KRIVELEVICH *et al*
- 85 Fourier analysis: Part I – Theory, ADRIAN CONSTANTIN
- 86 Dispersive partial differential equations, M. BURAK ERDOĞAN & NIKOLAOS TZIRAKIS
- 87 Riemann surfaces and algebraic curves, R. CAVALIERI & E. MILES
- 88 Groups, languages and automata, DEREK F. HOLT, SARAH REES & CLAAS E. RÖVER

Preface

This book explores connections between group theory and automata theory. We were motivated to write it by our observations of a great diversity of such connections; we see automata used to encode complexity, to recognise aspects of underlying geometry, to provide efficient algorithms for practical computation, and more.

The book is pitched at beginning graduate students, and at professional academic mathematicians who are not familiar with all aspects of these interconnected fields. It provides background in automata theory sufficient for its applications to group theory, and then gives up-to-date accounts of these various applications. We assume that the reader already has a basic knowledge of group theory, as provided in a standard undergraduate course, but we do not assume any previous knowledge of automata theory.

The groups that we consider are all finitely generated. An element of a group G is represented as a product of powers of elements of the generating set X , and hence as a string of symbols from $A := X \cup X^{-1}$, also called words. Many different strings may represent the same element. The group may be defined by a presentation; that is, by its generating set X together with a set R of relations, from which all equations in the group between strings can be derived. Alternatively, as for instance in the case of automata groups, G might be defined as a group of functions generated by the elements of X .

Certain sets of strings, also called languages, over A are naturally of interest. We study the word problem of the group G , namely the set $WP(G, A)$ of strings over A that represent the identity element. We define a language for G to be a language over A that maps onto G , and consider the language of all geodesics, and various languages that map bijectively to G . We also consider combings, defined to be group languages for which two words representing either the same element or elements that are adjacent in the Cayley graph fellow travel; that is, they are at a bounded distant apart throughout their length.

We consider an automaton to be a device for defining a, typically infinite, set L of strings over a finite alphabet, called the language of the automaton. Any string over the finite alphabet may be input to the automaton, and is then either accepted if it is in L , or rejected if it is not. We consider automata of varying degrees of complexity, ranging from finite state automata, which define regular languages, through pushdown automata, defining context-free languages, to Turing machines, which set the boundaries for algorithmic recognition of a language. In other words, we consider the full Chomsky hierarchy of formal languages and the associated models of computation.

Finite state automata were used by Thurston in his definition of automatic groups after he realised that both the fellow traveller property and the finiteness of the set of cone types that Cannon had identified in the fundamental groups of compact hyperbolic manifolds could be expressed in terms of regular languages. For automatic groups a regular combing can be found; use of the finite state automaton that defines this together with other automata that encode fellow travelling allows in particular a quadratic time solution to the word problem. Word-hyperbolic groups, as defined by Gromov, can be characterised by their possession of automatic structures of a particular type, leading to linear time solutions to the word problem.

For some groups the set of all geodesic words over A is a regular language. This is true for word-hyperbolic groups and abelian groups, with respect to any generating set, and for many other groups, including Coxeter groups, virtually free groups and Garside groups, for certain generating sets. Many of these groups are in fact automatic.

The position of a language in the Chomsky hierarchy can be used as a measure of its complexity. For example, the problem of deciding whether an input word w over A represents the identity of G (which, like the set it recognises, is called the word problem) can be solved by a terminating algorithm if and only if the set $WP(G, A)$ and its complement can be recognised by a Turing machine; that is, if and only if $WP(G, A)$ is recursive. We also present a proof of the well-known fact that finitely presented groups exist for which the word problem is not soluble; the proof of this result encodes the existence of Turing machines with non-recursive languages.

When the word problem is soluble, some connections can be made between the position of the language $WP(G, A)$ in the Chomsky hierarchy and the algebraic properties of the group. It is elementary to see that $WP(G, A)$ is regular if and only if G is finite, while a highly non-trivial result of Muller and Schupp shows that a group has context-free word problem if and only if it has a free subgroup of finite index.

Attaching an output-tape to an automaton extends it from a device that

defines a set of strings to a function from one set of strings to another. We call such a device a transducer, and show how transducers can be used to define groups. Among these groups are finitely generated infinite torsion groups, groups of intermediate growth, groups of non-uniform exponential growth, iterated monodromy groups of post-critically finite self-coverings of the Riemann sphere, counterexamples to the strong Atiyah conjecture, and many others. Our account is by no means complete, as it concentrates on introducing terminology, the exposition of some basic techniques and pointers to the literature.

There is a shorter book by Ian Chiswell [68] that covers some of the same material as we do, including groups with context-free word problem and an introduction to the theory of automatic groups. Our emphasis is on connections between group theory and formal language theory rather than computational complexity, but there is a significant overlap between these areas. We recommend also the article by Mark Sapir [226] for a survey of results concerning the time and space complexity of the fundamental decision problems in group theory.

Acknowledgements We are grateful to Professor Rick Thomas for numerous helpful and detailed discussions on the topics covered in this book. Some of the text in Section 1.2, and the statement and proofs of Propositions 3.4.9 and 3.4.10 were written originally by Rick.

We are grateful also to Professor Susan Hermiller and to an anonymous reviewer for a variety of helpful comments.

Contents

Preface *page ix*

PART ONE INTRODUCTION 1

1	Group theory	3
1.1	Introduction and basic notation	3
1.2	Generators, congruences and presentations	5
1.3	Decision problems	7
1.4	Subgroups and Schreier generators	8
1.5	Combining groups	10
1.6	Cayley graphs	20
1.7	Quasi-isometries	23
1.8	Ends of graphs and groups	25
1.9	Small cancellation	26
1.10	Some interesting families of groups	28
2	Formal languages and automata theory	36
2.1	Languages, automata and grammars	36
2.2	Types of automata	38
2.3	More on grammars	50
2.4	Syntactic monoids	51
2.5	Finite state automata, regular languages and grammars	52
2.6	Pushdown automata, context-free languages and grammars	60
2.7	Turing machines, recursively enumerable languages and grammars	81
2.8	Linearly bounded automata, context-sensitive languages and grammars	85

2.9	Turing machines and decidability	87
2.10	Automata with more than one input word	91
3	Introduction to the word problem	97
3.1	Definition of the word problem	97
3.2	Van Kampen diagrams	98
3.3	The Dehn function	102
3.4	The word problem as a formal language	105
3.5	Dehn presentations and Dehn algorithms	110
3.6	Filling functions	114
	 PART TWO FINITE STATE AUTOMATA AND GROUPS	 115
4	Rewriting systems	117
4.1	Rewriting systems in monoids and groups	117
4.2	The use of fsa in the reduction process	123
5	Automatic groups	125
5.1	Definition of automatic groups	126
5.2	Properties of automatic groups	129
5.3	Shortlex and geodesic structures	135
5.4	The construction of shortlex automatic structures	135
5.5	Examples of automatic groups	139
5.6	Closure properties	142
5.7	The falsification by fellow traveller property	143
5.8	Strongly geodesically automatic groups	145
5.9	Generalisations of automaticity	146
6	Hyperbolic groups	150
6.1	Hyperbolicity conditions	150
6.2	Hyperbolicity for geodesic metric spaces and groups	153
6.3	Thin bigons, biautomaticity and divergence	154
6.4	Hyperbolic groups have Dehn presentations	158
6.5	Groups with linear Dehn functions are hyperbolic	161
6.6	Equivalent definitions of hyperbolicity	164
6.7	Quasigeodesics	165
6.8	Further properties of hyperbolic groups	166
7	Geodesics	169
7.1	Introduction	169
7.2	Virtually abelian groups and relatively hyperbolic groups	170
7.3	Coxeter groups	173

7.4	Garside groups	176
7.5	Groups with geodesics lying in some subclass of <i>Reg</i>	182
7.6	Conjugacy geodesics	182
8	Subgroups and coset systems	184
8.1	Rational and quasiconvex subsets of groups	184
8.2	Automatic coset systems	189
9	Automata groups	194
9.1	Introducing permutational transducers	194
9.2	Automata groups	199
9.3	Groups of tree automorphisms	203
9.4	Dual automata	210
9.5	Free automata groups	212
9.6	Decision problems	213
9.7	Resolving famous problems	215
	 PART THREE THE WORD PROBLEM	 219
10	Solubility of the word problem	221
10.1	The Novikov–Boone theorem	221
10.2	Related results	226
11	Context-free and one-counter word problems	228
11.1	Groups with context-free word problem	228
11.2	Groups with one-counter word problem	232
12	Context-sensitive word problems	236
12.1	Lakin’s example	237
12.2	Some further examples	239
12.3	Filling length	240
12.4	Groups with linear filling length	242
13	Word problems in other language classes	249
13.1	Real-time word problems	249
13.2	Indexed word problems	254
13.3	Poly-context-free word problems	254
14	The co-word problem and the conjugacy problem	256
14.1	The co-word problem	256
14.2	Co-context-free groups	257
14.3	Indexed co-word problems	267
14.4	The conjugacy problem	268

<i>References</i>	270
<i>Index of Notation</i>	283
<i>Index of Names</i>	284
<i>Index of Topics and Terminology</i>	287

PART ONE

INTRODUCTION

