# Integration in Hilbert Space

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## Author's Preface

Integration in function spaces arose in probability theory when a general theory of random processes was constructed. Here credit is certainly due to N. Wiener, who constructed a measure in function space, integrals with respect to which express the mean value of functionals of Brownian motion trajectories. Brownian trajectories had previously been considered as merely physical (rather than mathematical) phenomena. A. N. Kolmogorov generalized Wiener's construction to allow one to establish the existence of a measure corresponding to an arbitrary random process. These investigations were the beginning of the development of the theory of stochastic processes. A considerable part of this theory involves the solution of problems in the theory of measures on function spaces in the specific language of stochastic processes. For example, finding the properties of sample functions is connected with the problem of the existence of a measure on some space; certain problems in statistics reduce to the calculation of the density of one measure w. r. t. another one, and the study of transformations of random processes leads to the study of transformations of function spaces with measure. One must note that the language of probability theory tends to obscure the results obtained in these areas for mathematicians working in other fields.

Another direction leading to the study of integrals in function space is the theory and application of differential equations. A. N. Kolmogorov has shown a connection between the solutions of second-order parabolic differential equations and the means of certain stochastic processes (i. e., integrals w. r. t. measures in function space). Essential progress was made here by M. Kac, who expressed the solution of the equation

$$u_t = u_{xx} + v\,u$$

by means of an integral w. r. t. Wiener measure.

However, the most significant step in this direction was taken by R. Feynman who put his "continual integral" (the Feynman integral) at the basis of the structure of quantum mechanics. In particular, with the help of such an integral he was able to express the solution of the Schrödinger equation. The Feynman integral differs from the Wiener integral in that there exists no completely-additive measure with respect to which it can be written, so that the problem of the existence of the Feynman integral is a difficult one and has not yet been solved. The mathematical aspects of the problem are treated in the survey by Gel'fand and Yaglom [1]. The continual integral was subsequently used to investigate evolution equations of order greater than two. Unfortunately, it turned out that no measures in function space correspond to such equations, but rather quasi-measures, i. e., signed, finitely-additive set functions of unbounded variation. The deepest results here have been obtained by Ju. L. Daleckii (see, for example, his survey [1]).

In this book we consider only measures on function spaces which are taken to be separable Hilbert space. This is due to the fact that although Hilbert space is not essential in many cases, a number of important problems have been solved only for Hilbert space. To the latter is related the problem of the existence of a measure. At the same time, constructions carried out in Hilbert space can often be easily generalized to rather general linear spaces. There are remarks on this in the notes collected at the end of the book. Some justification for restricting ourselves to Hilbert space is also provided by the fact that Hilbert space has so far been sufficient for applications.

The author has set himself the goal of an orderly presentation in measure-theoretic language of the basic ideas of the theory of measure and integration in Hilbert Space, including those which have heretofore been available only in the theory of stochastic processes,. To the most important questions we consider are related: 1) methods of defining a measure and conditions for its existence, 2) measurable functions on Hilbert space with measure, 3) the construction of systems of orthogonal functions, 4) the absolute continuity of measures and the calculation of the density of one measure with respect to another, 5) the theory of quasi-invariant measures, 6) transformations of measures under transformations of the space and 7) surface integrals and Green's formula in Hilbert space. A considerable part of this material is published here for the first time.

In bibliographic notes collected at the end of the book we have attempted to clarify the role played by various authors in the development of the above ideas.

A. V. Skorohod

## Translator's Preface

One of the satisfying aspects of mathematics is the ease (relative to other human pursuits) with which progress in certain of its tangled branches can be summarized and unified. That this is still not *Jedermanns Sache* is shown by the fact the subject of this book has waited fifty years for review and summary. To be sure, the introduction of Wiener's Differential Space in the early 20's was followed by a period in which it was viewed by many as a curiosity and neither understood nor appreciated. But the rapid development since 1923 of quantum physics and of the broad concept of "adaptive control" (two of the most exciting of the many directions taken by applied mathematics) and their increased (and occasionally incorrect) use of integration in function spaces has indicated that it is high time for such a book. The author is an expert on the subject whose various researches in the theory of stochastic processes have been closely connected with several areas of applied probability which are growing so rapidly that they often leave the rigorous justification of their techniques behind.

*Skorohod* has provided integration with respect to measures in function (Hilbert) space with a rigorous foundation in this important book. Through the good offices of Springer-Verlag and Nauka in Moscow I was able to begin translating it before its publication in the Soviet Union, which has advanced its availability in translation by several months.

I have corrected a number of misprints and typographical omissions and occasionally added what I believe to be a clarifying footnote. As usual, it was necessary to compile an index. Otherwise, this edition is the same as the original.

Thanks are due to Professor Skorohod for carefully reading the manuscript and thus clarifying many obscurities, and to my erstwhile companion M. for various favors.

Manchester, England, June 1974 K. Wickwire

# Introduction

The general theory of measures has been constructed for arbitrary measurable spaces, i.e., for sets on which a σ-algebra of measurable subsets has been selected. It can thus be shown that when the measurable space is linear and the σ-algebra is related in a certain way to the algebraic structure of the space, then no special theory is necessary. This occurs if we restrict ourselves to a finite-dimensional space with a σ-algebra of Borel sets. Of course, in this case as well there arise special problems connected, for example, with invariant measures. However, no special theory is required for their solution. The situation changes considerably if we go over to infinite-dimensional spaces. For many important problems it is still impossible to give solutions as simple as those in the finite-dimensional case. Here are two examples of such problems. The first of these is that of defining a measure. In the finite-dimensional case, it is sufficient to define a measure on all parallelepipeds with sides parallel to the coordinate axes; the values of the measure on these sets are defined by some (distribution) function, so that to each such function there corresponds a measure. In the infinite-dimensional case this does not hold: there need not correspond a measure to each distribution function. The existence problem for a measure is far from being solved for all spaces. The second problem concerns conditions for the absolute continuity of measures and the form of the corresponding density. In the finite-dimensional case it is solved by differentiating the distribution function. In the infinite-dimensional case, the solution of this problem, even for concrete measures, is not at all trivial.

Hilbert space is the simplest and most natural generalization of a finite-dimensional space; in it are manifested all of the difficulties connected with an infinite number of dimensions. At the same time, the theory for this space has been most completely developed, so that a coherent presentation of it is already possible. The present book is devoted to this theory.

We list briefly the basic problems which will be treated and also mention the principal results.

Chapter 1 is devoted to methods of defining measures. The notions of finite-dimensional distribution, weak distribution and characteristic functional are defined and the Minlos-Sazonov theorem is proved, which gives necessary and sufficient conditions for the existence of a measure with given weak distribution (or characteristic functional). In this chapter we also define the class of Gaussian measures, which is important — especially for the theory of probability.

In the second chapter we consider measurable functions, in particular, linear and polynomial functions. It is necessary to note that a peculiarity of infinite-dimensional spaces is also manifested here: in the finite-dimensional case a measurable polynomial is necessarily continuous, but in Hilbert space this is not so. With the aid of a certain procedure which we describe it is possible to reduce the investigation of polynomial functions to the study of linear ones on some other space. We also consider various systems of functions which are orthogonal w.r.t. a given measure.

In Chapter 3 we study general questions of absolute continuity of measures in Hilbert space. As a preliminary we define and prove the existence of conditional measures and we also prove theorems on the convergence of martingales and semi-martingales. The general assumptions are adapted to product measures, Gaussian measures and mixed measures, i.e., measures obtained by mixing measures depending on a parameter and integrated w.r.t. such a parameter. We remark that a large number of papers of a probability-theoretic or applied character have been devoted to the question of the absolute continuity and singularity of Gaussian measures.

Chapter 4 investigates admissible shifts (translations) of a measure, i.e., shifts transporting a measure into another one which is absolutely continuous w.r.t. the original. The structure of the set of admissible shifts is studied and a condition for the admissibility of a shift is found in terms of the derivative of a measure w.r.t. a given direction. A peculiarity of infinite-dimensional space is the lack of a Lebesgue measure (invariant w.r.t. a shift) and even of a measure for which all shifts are admissible. Thus, measures are of interest for which there exists a sufficiently rich set of admissible shifts, for example a linear set, dense in the whole space. Such measures are said to be quasi-invariant. We will give a complete description of these.

Finally, in Chapter 5 we generalize some formulas of classical analysis to the infinite-dimensional case. The first of these is the substitution formula for integrals. In the case where the integral is taken w.r.t. a Gaussian measure, this topic has been the subject of lively discussion in the literature of probability theory for more than twenty years. Another problem relates to the construction of a surface integral connected with a measure which is not concentrated on the surface (exactly such a situation holds in the evaluation of the Lebesgue area of a surface from the Lebesgue volume it encloses.) We obtain a generalized Gauss formula for the construction of such a surface integral.

The reader is expected to be acquainted with the basic theory of Hilbert space as well as that of measure and integration. In view of the large number of texts on these topics it will not be necessary to single out any for special recommendation; the author has endeavored to reduce to a minimum the use of definitions not given explicitly in the book as well as the number of unproved theorems. The notation is more or less customary making it unnecessary to offer any special explanations. References to the literature are not as a rule given in the text and are collected in notes at the end of the book.

# Contents

Introduction . . . . . . . . . . . . . . . . . . . . . . . . X

Chapter 1. Definition of a Measure in Hilbert Space . . . . . . 1

§ 1. Measurable Hilbert Spaces . . . . . . . . . . . . . 1
§ 2. Weak Distributions . . . . . . . . . . . . . . . . 3
§ 3. The Characteristic Functional. Moment Functionals . . 11
§ 4. The Minlos-Sazonov Theorem . . . . . . . . . . . . 15
§ 5. Gaussian Measures . . . . . . . . . . . . . . . . 18
§ 6. Generalized Measures in Hilbert Space . . . . . . . . 23

Chapter 2. Measurable Functions on Hilbert Space . . . . . . 27

§ 7. Measurable Linear Functionals . . . . . . . . . . . 27
§ 8. Measurable Linear Operators . . . . . . . . . . . . 31
§ 9. Measurable Polynomial Functions . . . . . . . . . . 36
§ 10. Square-integrable Polynomials . . . . . . . . . . . 43
§ 11. Orthogonal Systems of Polynomials . . . . . . . . . 48
§ 12. Polynomials Orthogonal with Respect to a Weight Function . . . . . . . . . . . . . . . . . . . . . . 53

Chapter 3. Absolute Continuity of Measures . . . . . . . . . 57

§ 13. The Radon-Nikodym Theorem. Conditional Measures 57
§ 14. Martingales and Semi-Martingales . . . . . . . . . . 63
§ 15. General Conditions for Absolute Continuity . . . . . . 69
§ 16. Absolute Continuity of Product Measures . . . . . . . 78
§ 17. Absolute Continuity of Gaussian Measures . . . . . . 85
§ 18. Absolute Continuity of Mixed Measures . . . . . . . . 95

Chapter 4. Admissible Shifts and Quasi-invariant Measures . . . 102

§ 19. Admissible Shifts of Measures . . . . . . . . . . . 102
§ 20. Admissible Directions . . . . . . . . . . . . . . . 110

§ 21. Differentiation of Measures w.r.t. a Direction . . . . . . 114
§ 22. An Admissibility Condition for Shifts . . . . . . . . . 122
§ 23. Quasi-invariant Measures . . . . . . . . . . . . . . . 128

Chapter 5. Some Questions of Analysis in Hilbert Space . . . . 140

§ 24. The Substitution Formula and Absolute Continuity . . . 140
§ 25. Linear Transformations . . . . . . . . . . . . . . . 143
§ 26. Absolute Continuity of Measures under Nonlinear Transformations . . . . . . . . . . . . . . . . . . . . . 152
§ 27. Surface Integrals . . . . . . . . . . . . . . . . . . 157
§ 28. Gauss' Formula . . . . . . . . . . . . . . . . . . . 163

Bibliographic Notes . . . . . . . . . . . . . . . . . . . . 169
Bibliography . . . . . . . . . . . . . . . . . . . . . . . 172
Index . . . . . . . . . . . . . . . . . . . . . . . . . . . 175

# Chapter I. Definition of a Measure in Hilbert Space

## § 1. Measurable Hilbert Spaces

Let $X$ be a real separable Hilbert space with elements $x, y, z$, etc. Real numbers will be designated by small Greek letters; $\alpha x + \beta y$ and $(x, y)$ will denote, as usual, the operations of multiplication of a vector (element of $X$) by a scalar, vector addition and the scalar product of vectors. The norm of a vector will be designated by

$$|x| = \sqrt{(x, x)}\,.$$

Subsets of $X$ will be denoted by large Latin letters, classes of subsets by large Gothic letters. A class of sets $\mathfrak{A}$ in which we allow the operations of set difference, union and intersection is called a ring. A ring of sets $\mathfrak{A}$ containing $X$ as an element is called an algebra. An algebra of sets in which the union operation can be applied countably many times is called a $\sigma$-algebra.

In the sequel the letter $\mathfrak{B}$ will denote the $\sigma$-algebra of Borel sets in $X$, i.e., the minimal $\sigma$-algebra of subsets of $X$ containing all open sets (because of the separability of the space it is sufficient that the $\sigma$-algebra contain all spheres.) The measurable space $(X, \mathfrak{B})$, i.e., the set $X$ with $\sigma$-algebra of measurable sets will be called a measurable Hilbert space.

We will study finite measures $\mu$ defined on $(X, \mathfrak{B})$. In this connection it is often convenient to assume that the measure $\mu$ is normalized, i.e., $\mu(X) = 1$. The primary aim of this chapter is to propose a method of constructing measures on $(X, \mathfrak{B})$. This method is essentially equivalent to the method of Lebesgue for constructing the measure on the line: first the measure is constructed on a certain class of elementary sets and then extended to the minimal $\sigma$-algebra containing these sets and completed. The latter operation (completion) will not be carried out since we only need the measure on $\mathfrak{B}$.

We will investigate certain classes of "simple" sets on which the values of the measure will be assigned. Let $L$ be a finite-dimensional subspace of $X$, $P_L$ the orthogonal projection operator on $L$ and $A$ a Borel set from $L$. A set of the form

$$\{x\colon P_L x \in A\}$$

will be called a cylinder set and the set $A$ is called its base (we also say that this is a cylinder set with base in $L$). The class of all cylinder sets with bases in $L$ is obviously a $\sigma$-algebra which we will write as $\mathfrak{B}^L$, whereby $\mathfrak{B}^L \subset \mathfrak{B}$. The union of all $\sigma$-algebras $\mathfrak{B}^L$ is an algebra. Indeed, if $A_1$ and $A_2$ belong to $\mathfrak{B}^{L_1}$ and $\mathfrak{B}^{L_2}$, then choosing $L = L_1 + L_2$ (the sum of the subspaces), we get

$$A_i \in \mathfrak{B}^L, i = 1, 2;\quad A_1 \cup A_2 \in \mathfrak{B}^L;\quad A_1 \cap A_2 \in \mathfrak{B}^L;\quad A_1 - A_2 \in \mathfrak{B}^L .$$

We denote this algebra of sets by $\mathfrak{B}_0$. It is called the algebra of cylinder sets. Sets from $\mathfrak{B}_0$ are also considered as "elementary" for the construction of a measure on $(X, \mathfrak{B})$. To convince ourselves that the value of the measure on $\mathfrak{B}_0$ uniquely defines that on $\mathfrak{B}$ it is necessary to show that the $\sigma$-closure of $\mathfrak{B}_0$ contains $\mathfrak{B}$ (i.e., $\mathfrak{B}$ is the smallest $\sigma$-algebra containing $\mathfrak{B}_0$). We will show this below and immediately note that the algebra $\mathfrak{B}_0$ still contains too many sets ($X$ has too many finite-dimensional subspaces). It turns out that to define the measure it is sufficient to have a certain chain of increasing subspaces $L_n \subset L_{n+1}$ for which $\bigcup_n L_n$ is dense in $X$. Then $\bigcup_n \mathfrak{B}^{L_n} = \mathfrak{B}_0'$ will also be an algebra of sets. Since $\mathfrak{B}_0' \subset \mathfrak{B}_0$, it follows from the fact that the $\sigma$-closure of $\mathfrak{B}_0'$ coincides with $\mathfrak{B}$, that the $\sigma$-closure of $\mathfrak{B}_0$ coincides with $\mathfrak{B}$. Let us prove the first assertion. It is sufficient to show that the $\sigma$-closure of $\mathfrak{B}_0'$ contains any closed sphere $S$ in $X$ since an open sphere can be represented as a countable sum of increasing closed spheres. Set

$$S = \{x\colon |x - a| \le \varrho\} .$$

Denote by $P_n$ the operator projecting onto $L_n$ and put

$$S_n = \{x\colon |P_n x - P_n a| \le \varrho\} .$$

The set $S_n$ belongs to $\mathfrak{B}^{L_n}$ and $S_n \supset S$. We will prove that

$$S = \bigcap_n S_n . \tag{1}$$

In fact, if $y \notin S$, then $|y - a| = \varrho + \delta$, $\delta > 0$.
But

$$\lim_{n\to\infty} P_n(y - a) = y - a$$

(in the sense of convergence in $X$). Thus, $|P_n(y - a)| \to |y - a|$ which means that for large enough $n$ $|P_n(y - a)| = |P_n y - P_n a| > \varrho$, $y \notin S_n$. We have proved (1).

The advantage of the algebra $\mathfrak{B}_0'$ lies in the fact that it is the union of countably many $\sigma$-algebras of the form $\mathfrak{B}^L$. We now note that sets of the algebra $\mathfrak{B}^L$ are completely determined by Borel sets of the finite-dimensional Euclidean space $L$ ($L$ will be such a space if it is considered by itself and not as a subset of $X$). If $L$ is an $n$-dimensional space, $\{e_k; k = 1, \ldots, n\}$ is an orthonormalized basis in $L$ and $x = \sum \xi_k e_k$ is a decomposition of an arbitrary $x \in L$ w.r.t. this basis, then finite sums of sets of the form

$$\{x: \alpha_k \leq \xi_k \leq \beta_k; k = 1, \ldots, n; -\infty \leq \alpha_k < \beta_k \leq \infty\}$$

(these sets are called *rectangles*) generate an algebra of subsets of $L$, whose $\sigma$-closure coincides with the $\sigma$-algebra $\mathfrak{B}_L$ of Borel sets of $L$. The same property will be possessed by the algebra $\mathfrak{A}_L$, generated by rectangles with rational $\alpha_k$ and $\beta_k$. Let $\mathfrak{A}^L$ be the algebra of cylinder sets with bases in $\mathfrak{A}_L$. Then the $\sigma$-closure of $\mathfrak{A}^L$ coincides with $\mathfrak{B}^L$. Consequently, for an increasing sequence of subspaces $L_n$ for which $\bigcup_n L_n$ is dense in $X$, the algebra

$$\mathfrak{A}'_0 = \bigcup_n \mathfrak{A}^{L_n}$$

(we assume that bases in the $L_n$ are chosen in a compatible way, i.e., the basis in $L_{n+1}$ is obtained from that in $L_n$ by adding the basis from the orthogonal complement to $L_n$ in $L_{n+1}$; in this case $\mathfrak{A}'_0$ will actually be a set algebra since $\mathfrak{A}^{L_n} \subset \mathfrak{A}^{L_{n+1}}$) is such that its $\sigma$-closure contains $\mathfrak{B}$. Each of the algebras $\mathfrak{A}^{L_n}$ contains only a countable number of sets, which implies that $\mathfrak{A}'_0$ also contains a countable number of sets. We recall that the $\sigma$-algebra obtained from the $\sigma$-closure of a denumerable algebra of sets is said to be separable. By the same token, we have established that the $\sigma$-algebra $\mathfrak{B}$ is separable.

The algebras $\mathfrak{B}'_0$ and $\mathfrak{A}'_0$ make the measure problem easier since they contain fewer elementary sets. However, they depend on a certain set of finite-dimensional subspaces (and even on the bases in these subspaces), so that the definition of measures by means of these algebras possesses a non-invariant character. Hence, in those cases where it is necessary to describe the invariant properties of measures we will use the algebra $\mathfrak{B}_0$.

## § 2. Weak Distributions

Let $\mu$ be some normalized measure on $(X, \mathfrak{B})$. For each finite-dimensional subspace $L$ of the space $X$ we can consider the restriction of this measure to $\mathfrak{B}^L$. Now define the measure $\mu_L$ on the $\sigma$-algebra $\mathfrak{B}_L$ of Borel sets of $L$ as follows:

$$\mu_L(A) = \mu(\{x: P_L x \in A\}),$$

where $P_L$ is projection onto the subspace $L$. Then the fact that $\mu_L$ is a measure follows from the fact that for a sequence of non-overlapping sets $A_n$ of $\mathfrak{B}_L$, the sets

$$P_L^{-1}\left(\bigcup_n A_n\right) = \bigcup_n P_L^{-1}(A_n),\ P_L^{-1}(A_n)$$

are also non-overlapping (here $P_L^{-1}(A)$ is the inverse image of the set $A$ under the projection operator $P_L$, i.e., $P_L^{-1}(A) = \{x: P_L x \in A\}$). The measure $\mu_L$ is called the projection of the measure $\mu$ onto the subspace $L$.

Hence, with each measure $\mu$ we can associate the set of its projections $\{\mu_L\}$ on finite-dimensional subspaces of $X$. Obviously, knowing $\mu_L$, one can determine $\mu$ on $\mathfrak{B}^L$. Hence, knowing $\mu_{L_n}$ for a sequence $L_n$ of linear subspaces for which $L_n \subset L_{n+1}$ and $\bigcup_n L_n$ is dense in $X$, we can define $\mu$ on $\bigcup_n \mathfrak{B}^{L_n}$ and since the $\sigma$-closure of this algebra coincides with $\mathfrak{B}$ we can define $\mu$ on $\mathfrak{B}$ by the same token. This means that knowing $\{\mu_L\}$ or $\{\mu_{L_n}\}$, where $L_n$ is the indicated sequence of subspaces, we can uniquely retrieve the measure.

The totality of all projections of a given measure are called the *finite-dimensional distributions* of the measure. The measures $\mu_{L_n}$ being projections of the same measure are compatible in a certain sense for different $n$. This compatibility condition follows from the fact that the base of a cylinder set is chosen non-uniquely. Let $L_1 \subset L_2$ and $A \in \mathfrak{B}_{L_1}$. Then the set $P_{L_1}^{-1}(A)$ can also be written as $P_{L_2}^{-1}(A_2)$, where $A_2 \in \mathfrak{B}_{L_2}$ is defined by the equality

$$A_2 = \{x: x \in L_2,\ P_{L_1} x \in A\}.$$

Since $P_{L_1}^{-1}(A) = P_{L_2}^{-1}(A_2)$, we have

$$\mu_{L_1}(A) = \mu\left(P_{L_1}^{-1}(A)\right) = \mu\left(P_{L_2}^{-1}(A_2)\right) = \mu_{L_2}(A_2)$$

Since $A_2 = P_{L_1}^{-1}(A) \cap L_2$, the compatibility condition can be written in the following form: for all $L_1 \subset L_2$ and $A \in \mathfrak{B}_{B_1}$

$$\mu_{L_1}(A) = \mu_{L_2}\left(P_{L_1}^{-1}(A) \cap L_2\right). \tag{1}$$

The family of measures $\{\mu_L\}$, defined for all finite-dimensional subspaces $L$ and satisfying the compatibility condition (1) is called a *weak distribution*.

If $L_n$ is a sequence of subspaces, $L_n \subset L_{n+1}$, $\bigcup L_n$ dense in $X$ and $\mu_{L_n}$ is a sequence of measures on $\mathfrak{B}_{L_n}$ satisfying the compatibility condition

$$\mu_{L_n}(A) = \mu_{L_{n+1}}\left(P_{L_n}^{-1}(A) \cap L_{n+1}\right),$$

then the sequence $\{\mu_{L_n}\}$ is called a *sequence of finite-dimensional distributions*.

From the results above it follows that to each measure on $(X, \mathfrak{B})$ there corresponds some weak distribution, and different weak distributions correspond to different measures. The problem of defining a measure with the help of weak distributions could now be solved quite easily if to each weak distribution therecorresponded some measure on $(X, \mathfrak{B})$. Unfortunately this is not the case. We will now derive conditions which must be satisfied by a weak distribution in order that some measure correspond to it.

**Lemma 1.** *Let $S_\varrho$ be a sphere of radius $\varrho$: $S_\varrho = \{x: |x| \leq \varrho\}$. The weak distribution $\{\mu_L\}$ will be generated by some measure $\mu$ on $(X, \mathfrak{B})$ iff, for every $\varepsilon > 0$ there exists an $\eta > 0$ such that for all L*

$$\mu_L (S_\varrho \cap L) \geq 1 - \varepsilon \quad \text{when} \quad \varrho > \eta .$$

*Proof. Necessity.* If $\{\mu_L\}$ is generated by the measure $\mu$, then choosing $\eta$ such that $\mu(S_\eta) > 1 - \varepsilon$ (this is possible since $\lim\limits_{\eta \to +\infty} \mu(S_\eta) = \mu(X) = 1$), we obtain

$$\mu_L (S_\varrho \cap L) = \mu \left(P_L^{-1} (S_\varrho \cap L)\right) \geq \mu(S_\varrho) \geq \mu(S_\eta) > 1 - \varepsilon .$$

The proof of *sufficiency* is more difficult. We define on the algebra $\mathfrak{B}_0 = \bigcup\limits_L \mathfrak{B}^L$ a finitely additive function $\mu$ be means of

$$\mu(A) = \mu_L(A) , \quad A \subset \mathfrak{B}^L .$$

To convince ourselves that $\mu$ can be extended to a measure defined on $(X, \mathfrak{B})$, it is sufficient to show that $\mu$ is continuous on $\mathfrak{B}_0$, i.e., that for an arbitrary sequence of sets $A_n \in \mathfrak{B}_0$ for which $A_n \supset A_{n+1}$ and $\bigcap\limits_n A_n = \phi$ ($\phi$ is the empty set),

$$\lim_{n\to\infty} \mu(A_n) = 0 . \tag{2}$$

Let $A_n$ be a cylinder set with base in $L_n$ and $L_n \subset L_{n+1}$. Let $B_n \subset L_n$ be the base of $A_n$. We remark that it is sufficient to prove (2) merely for sets with closed bases. Indeed, choosing closed sets $C_n \subset B_n$ such that $\mu_{L_n}(B_n - C_n) < \varepsilon_n$ and then taking

$$D_n = \bigcap_{m=1}^{n} \{x: P_{L_m} x \in C_m\} \cap L_n ,$$

we obtain closed sets for which

$$\mu_{L_n}(B_n - D_n) \leq \sum_{m=1}^{n} \mu_{L_m} (B_m - C_m) \leq \sum_{m=1}^{n} \varepsilon_m$$

Hence, if $A_n' = P_{L_n}^{-1}(D_n)$, then

$$A_{n+1}' \subset A_n', \ \bigcap_n A_n' = \bigcap_n A_n \quad \text{and} \quad \mu(A_n) \leq \mu(A_n') + \sum_{m=1}^{n} \varepsilon_m$$

If for cylinder sets with closed bases $\lim_{n\to\infty} \mu(A_n') = 0$, then since $\sum_{m=1}^{\infty} \varepsilon_n$ can be taken as suitably small, (2) is also fulfilled. Hence, it will be assumed that the $B_n$ are closed sets. Then the $A_n$ will be weakly closed sets (if $x_k \xrightarrow{w.} x$ and $x_k \in A_n$, then $x \in A_n$). For all $\varrho$ the set $S_\varrho$ is also weakly closed and weakly compact. Since

$$S_\varrho \cap \left[\bigcap_{n=1}^{\infty} A_n\right] = \phi ,$$

$\bigcup_{n=1}^{\infty} [S_\varrho \cap A_n] = \phi$ and since the sets $S_\varrho \cap A_n$ are weakly closed and weakly compact and $S_\varrho \cap A_n \supset S_\varrho \cap A_{n+1}$, for some $n$ $S_\varrho \cap A_n = \phi$. This means that

$$\mu(A_n) = \mu_{L_n}(A_n) \le \mu_{L_n}(L_n) - \mu_{L_n}(L_n \cap S_\varrho) \le \varepsilon ,$$

provided that $\varrho > \eta$ ($\eta$ and $\varepsilon$ are defined in the lemma). From the arbitrariness of $\varepsilon > 0$ there follows (2). □

Remark. Let $L_n$ be a sequence of finite-dimensional subspaces for which $L_n \subset L_{n+1}$ and $\bigcup L_n$ is dense in $X$, and $\mu_{L_n}$ a sequence of finite-dimensional distributions. This sequence generates some measure iff for arbitrary $\varepsilon > 0$ there exists an $\eta > 0$ such that for all $n$ $\mu_{L_n}(S_\varrho \cap L_n) \ge 1 - \varepsilon$ when $\varrho > \eta$. The proof of this fact is carried out as in the proof of Lemma 1.

It is interesting to note that some classes of functions can also be integrated w.r.t. a weak distribution. To these functions, for example, are related the "cylinder functions" defined below. The function $\varphi(x)$ is called a *cylinder function* if for some finite-dimensional subspace $L$ it is $\mathfrak{B}^L$-measurable. In other words, every cylinder function $\varphi(x)$ has the form

$$\varphi(x) = \varphi_L(P_L x) , \tag{3}$$

where $\varphi_L$ is some $\mathfrak{B}_L$-measurable function defined on $L$, and $L$ is a finite-dimensional subspace of $X$. For each nonnegative cylinder function $\varphi(x)$ we define the "integral" w.r.t. the weak distribution $\{\mu_L\}$ which will be denoted by $\mu_*$ to distinguish it from a measure. This integral is defined by the relation

$$\int \varphi(x) \mu_*(dx) = \int \varphi_L(x) \mu_L(dx) , \tag{4}$$

where $\varphi_L$ is as in the representation (3). Since (3) is not unique, we must show that

$$\int \varphi_L(x) \mu_L(dx)$$

does not depend on the choice of $L$. Let $L_1 \subset L_2$ and

$$\varphi(x) = \varphi_{L_2}(P_{L_2} x) = \varphi_{L_1}(P_{L_1} x) .$$