Cambridge studies in advanced mathematics 79

# Soliton Equations and Their Algebro-Geometric Solutions

## Volume I: (1 + 1)- Dimensional Continuous Models

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HELGE HOLDEN

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CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK
40 West 20th Street, New York, NY 10011-4211, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

http://www.cambridge.org



First published 2003

Printed in the United Kingdom at the University Press, Cambridge

*Typeface* Times 10/13 pt. *System* LaTeX 2$_\varepsilon$ [TB]

*A catalog record for this book is available from the British Library.*

*Library of Congress Cataloging in Publication Data*
Gesztesy, Fritz, 1953–
Soliton equations and their algebro-geometric solutions / Fritz Gesztesy,
Helge Holden.
p. cm. – (Cambridge studies in advanced mathematics ; 79)
Includes bibliographical references and index.
Contents: v. 1. (1 + 1)-dimensional continuous models
ISBN 0-521-75307-4 (v. 1)
1. Differential equations, Nonlinear – Numerical solutions. 2. Solitons.
I. Holden, H. (Helge), 1956– II. Title. III. Series.
QC20.7.D5 G47 2003
530.15′5355 – dc21 2002074069

ISBN 0 521 75307 4 hardback

# Acknowledgments

... and the manuscript was becoming an albatross about my neck.
There were two possibilities: to forget about it completely,
or to publish it as it stood; and I preferred the second.

*Robert P. Langlands*[1]

This monograph has grown out of work we have done over the past 15 years within the area of completely integrable nonlinear partial differential equations. Our starting point has been that of mathematical analysis with emphasis on spectral theoretic techniques. We have made every effort to be explicit and as detailed as possible in the presentation of results. Consequently, this volume has acquired a somewhat technical appearance. However, our experience in the extensive study of an area, especially one in which the notion of exactly solvable models dominates and hence explicit formulas abound, is that there can never be too many details — mathematics is not a spectator sport.

We are indebted to many co-workers in this endeavor for the joy of collaboration, among them, especially, Wolfgang Bulla, Ronnie Dickson, Victor Enol'skii, Ratnam Ratnaseelan, Walter Renger, Barry Simon, Wilhelm Sticka, Gerald Teschl, Karl Unterkofler, Rudi Weikard, and Zhongxin Zhao.

Special and heartfelt thanks are due to Harald Hanche-Olsen and Karl Unterkofler. Their around-the-clock unselfish assistance with all aspects, both mathematical and TEXnical, of an enterprise of this magnitude, has been of invaluable help to us. Thanks a lot, Harald and Karl!

We are grateful to all the individuals who read portions of the manuscript and supplied us with valuable comments and criticism. In particular, we would like to thank Wolfgang Bulla, Steve Clark, Harald Hanche-Olsen, Emma Previato, Gerald Teschl, Karl Unterkofler, and Rudi Weikard for their efforts to help us improve the presentation.

[1] *On the Functional Equations Satisfied by Eisenstein Series*, Lecture Notes in Mathematics, **544**, Springer, Berlin, 1976, p. III.

We have established a Web page with URL

`www.math.ntnu.no/~holden/solitons`

where we intend to keep an updated list of typographical errors for the benefit of the reader. We encourage the reader to send comments, corrections, etc., to the authors.

F.G. is indebted to Sergio Albeverio, Margarida de Faria, Gisèle Ruiz Goldstein, Jerry Goldstein, and Ludwig Streit for encouragement and support. Moreover, he thanks Gisèle and Jerry as well as Margarida and Ludwig for kind opportunities to lecture on various parts of the material in this volume at LSU, Baton Rouge, Louisiana, USA (April 1996) and CCM, Universidade da Madeira, Portugal (August 2001), respectively.

Research in this area has been funded in part by the University of Missouri Research Board (grant RB-97-086), the Norwegian Academy of Science and Letters, and last, but certainly not least, the Research Council of Norway.

July 2002

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# Contents

# Introduction

> It often happens that the understanding of the mathematical nature of an equation is impossible without a detailed understanding of its solutions.
>
> *Freeman J. Dyson*

**Background:** The discovery of solitary waves of translation goes back to Scott Russell in 1834, and during the remaining part of the 19th century the true nature of these waves remained controversial. It was only with the derivation by Korteweg and de Vries in 1895 of what is now called the Korteweg–de Vries (KdV) equation, that the one-soliton solution and hence the concept of solitary waves was put on a firm basis.[1] An extraordinary series of events took place around 1965 when Kruskal and Zabusky, while analyzing the numerical results of Fermi, Pasta, and Ulam on heat conductivity in solids, discovered that pulselike solitary wave solutions of the KdV equation, for which the name "solitons" was coined, interact elastically. This was followed by the 1967 discovery of Gardner, Greene, Kruskal, and Miura that the inverse scattering method allows one to solve initial value problems for the KdV equation with sufficiently fast-decaying initial data. Soon thereafter, in 1968, Lax found a new explanation of the isospectral nature of KdV solutions using the concept of Lax pairs and introduced a whole hierarchy of KdV equations. Subsequently, in the early 1970s, Zakharov and Shabat (ZS), and Ablowitz, Kaup, Newell, and Segur (AKNS) extended the inverse scattering method to a wide class of nonlinear partial differential equations of relevance in various scientific contexts ranging from nonlinear optics to condensed matter physics and elementary particle physics. In particular, solitons found numerous applications in classical and quantum field theory and in connection with optical communication devices.

Another decisive step forward in the development of completely integrable soliton equations was taken around 1974. Prior to that period, inverse spectral

[1] With hindsight, though, it is now clear that other researchers, such as Boussinesq, derived the KdV equation and its one-soliton solution prior to 1895, as described in the notes to Section 1.1.

methods in the context of nonlinear evolution equations had been restricted to spatially decaying solutions. In 1974–75, the arsenal of inverse spectral methods was extended considerably in scope to include periodic and certain classes of quasi-periodic and almost periodic KdV solutions. This new approach to constructing solutions of integrable nonlinear evolution equations, partly based on inverse spectral theory and partly relying on algebro-geometric methods developed by pioneers such as Dubrovin, Flaschka, Its, Krichever, Lax, Marchenko, Matveev, McKean, Novikov, van Moerbeke – to name just a few – was followed by very rapid development in the field. Within a few years of intense activity worldwide, the landscape of integrable systems was changed forever. By the early 1980s the theory was extended to a large class of nonlinear (including some multi-dimensional) evolution equations beyond the KdV equation, and the explicit theta function representations of quasi-periodic solutions of integrable equations (including, e.g., soliton solutions as special limiting cases) had introduced new algebro-geometric techniques into this area of nonlinear partial differential equations. Subsequently, this led to several new and deep results in nonlinear partial differential equations as well as in algebraic geometry (such as a solution of Schottky's problem).

Our series of monographs is devoted to this area of algebro-geometric solutions of hierarchies of soliton equations.

**Scope:** We aim for an elementary, yet self-contained and precise, presentation of hierarchies of integrable soliton equations and their algebro-geometric solutions. Our point of view is predominantly influenced by analytical methods, especially by spectral theoretic techniques. We hope this will make the presentation accessible and attractive to analysts working outside the traditional areas associated with soliton equations. Central to our approach is a simultaneous construction of all algebro-geometric solutions and their theta function representation of a given hierarchy. In this volume we focus on some of the key hierarchies in $(1+1)$-dimensions associated with continuous integrable models such as the Korteweg–de Vries hierarchy (KdV), the combined sine–Gordon modified Korteweg–de Vries hierarchy (sGmKdV), the Ablowitz–Kaup–Newell–Segur hierarchy[1] (AKNS), the classical massive Thirring system (Th), and the Camassa–Holm hierarchy (CH). The key equations defining the corresponding hierarchies read

$$
\begin{aligned}
&\text{KdV:} && u_t + \tfrac{1}{4}u_{xxx} - \tfrac{3}{2}uu_x = 0,\\
&\text{sGmKdV:} && u_{xt} - \sin(u) = 0,\\
&\text{AKNS:} && \begin{pmatrix} p_t + \frac{i}{2}p_{xx} - ip^2q \\ q_t - \frac{i}{2}q_{xx} + ipq^2 \end{pmatrix} = 0,
\end{aligned}
\qquad (0.1)
$$

[1] Using the gauge equivalence of the AKNS hierarchy and classical Boussinesq hierarchy, we also treat the latter.

$$\text{Th:} \qquad \begin{pmatrix} -iu_x + 2v + 2vv^*u \\ iu_x^* + 2v^* + 2vv^*u^* \\ -iv_t + 2u + 2uu^*v \\ iv_t^* + 2u^* + 2uu^*v^* \end{pmatrix} = 0,$$

$$\text{CH:} \qquad 4u_t - u_{xxt} - 2uu_{xxx} - 4u_xu_{xx} + 24uu_x = 0.$$

Our principal goal in this monograph is the construction of algebro-geometric solutions of the hierarchies associated with the equations listed in (0.1). Interest in the class of algebro-geometric solutions can be motivated in a variety of ways: It represents a natural extension of the classes of soliton and rational solutions, and similar to these, its elements can still be regarded as explicit solutions of the nonlinear integrable evolution equation in question (even though their complexity considerably increases compared with soliton solutions due to the underlying analysis on compact Riemann surfaces). Moreover, algebro-geometric solutions can be used to approximate more general solutions (such as almost periodic ones), although this is not a topic pursued in this monograph. Here we primarily focus on the construction of explicit solutions in terms of certain algebro-geometric data on a compact Riemann surface and their representation in terms of theta functions. For instance, in KdV-type contexts, solitons arise as the special case of solutions corresponding to an underlying singular hyperelliptic curve obtained by confluence of two or more branch points, and rational solutions correspond to a further singularization of the original curve. In either case, the theta function associated with the underlying algebraic curve degenerates into appropriate determinants with exponential, respectively, rational entries.

We use basic techniques from the theory of differential equations, some spectral analysis, and elements of algebraic geometry (most notably, the basic theory of compact Riemann surfaces). In particular, we do not employ more advanced tools such as loop groups, Grassmanians, Lie algebraic considerations, formal pseudo-differential expressions, etc. However, occasionally we bridge the gap to spectral theory and its vicinity and include some finer points of the basic formalism often omitted in this context. Thus, this volume strays off the mainstream, but we hope it appeals to spectral theorists and their kin and convinces them of the beauty of the subject. In particular, we hope a reader interested in quickly penetrating to the fundamentals of the algebro-geometric approach of constructing solutions of hierarchies of completely integrable evolution equations will not be disappointed.

Completely integrable systems, and especially nonlinear evolution equations of soliton-type, are an integral part of modern mathematical and theoretical physics with far-reaching implications from pure mathematics to the applied sciences. We intend to contribute to the dissemination of some of the beautiful techniques applied in this area.

**Contents:** In the present volume we provide an effective approach to the construction of algebro-geometric solutions of certain completely integrable nonlinear evolution equations by developing a technique that simultaneously applies to all equations of the hierarchy in question.

Starting with a specific integrable partial differential equation, one can build an infinite sequence of higher-order partial differential equations, the so-called hierarchy of the original soliton equation, by developing an explicit recursive formalism that reduces the construction of the entire hierarchy to elementary manipulations with polynomials and defines the associated Lax pairs or zero-curvature equations. Using this recursive polynomial formalism, we simultaneously construct algebro-geometric solutions for the entire hierarchy of soliton equations at hand. On a more technical level, our point of departure for the construction of algebro-geometric solutions is not directly based on Baker–Akhiezer functions and axiomatizations of algebro-geometric data but rather on Dubrovin-type equations, trace formulas, and a canonical meromorphic function $\phi$ on the underlying hyperelliptic Riemann surface $\mathcal{K}_n$ of genus $n \in \mathbb{N}$. More precisely, this fundamental meromorphic function $\phi$ carries the spectral information of the underlying Lax operator (such as the Schrödinger and Dirac operators in the KdV and AKNS contexts) and in many instances represents a direct generalization of the Weyl–Titchmarsh $m$-function, a fundamental device in the spectral theory of ordinary differential operators. Riccati-type differential equations satisfied by $\phi$ separately in the space and time variables then govern the time evolutions of all quantities of interest (such as that of the associated Baker–Akhiezer vector). The basic meromorphic function $\phi$ on $\mathcal{K}_n$ is then linked with solutions of equations of the underlying hierarchy via trace formulas and Dubrovin-type equations for (projections of) the pole divisor of $\phi$. Subsequently, the Riemann theta function representation of $\phi$ is then obtained more or less simultaneously with those of the Baker–Akhiezer vector and the algebro-geometric solutions of the (stationary or time-dependent) equations of the hierarchy of evolution equations. This concisely summarizes our approach to all the $(1+1)$-dimensional, continuous integrable models discussed in this volume.

In the following we will detail this verbal description of our approach to algebro-geometric solutions of integrable hierarchies with the help of the KdV hierarchy. The latter consists of a sequence of nonlinear evolution equations for a function $u = u(x,t)$, the most prominent element of which, the KdV equation itself, is given by

$$u_t + \tfrac{1}{4}u_{xxx} - \tfrac{3}{2}uu_x = 0. \tag{0.2}$$

The KdV hierarchy is the simplest of all the hierarchies of nonlinear evolution equations studied in this volume, but the same strategy, with modifications to be discussed in the individual chapters, applies to all integrable systems treated in this monograph and is in fact typical for all $(1+1)$-dimensional integrable hierarchies of soliton equations.

A discussion of the KdV case then proceeds as follows.[1] In order to define the Lax pairs and zero-curvature pairs for the KdV hierarchy, one assumes $u$ to be a smooth function on $\mathbb{R}$ (or meromorphic in $\mathbb{C}$) in the stationary context or a smooth function on $\mathbb{R}^2$ in the time-dependent case, and one introduces the recursion relation for some functions $f_\ell$ of $u$ by

$$f_0 = 1, \quad f_{\ell,x} = -(1/4)f_{\ell-1,xxx} + uf_{\ell-1,x} + (1/2)u_x f_{\ell-1}, \quad \ell \in \mathbb{N}. \quad (0.3)$$

Given the recursively defined sequence $\{f_\ell\}_{\ell\in\mathbb{N}_0}$ (whose elements turn out to be differential polynomials with respect to $u$ defined up to certain integration constants) one defines the Lax pair of the KdV hierarchy by

$$L = -\frac{d^2}{dx^2} + u, \quad (0.4)$$

$$P_{2n+1} = \sum_{\ell=0}^{n} \left( f_{n-\ell}\frac{d}{dx} - \frac{1}{2} f_{n-\ell,x} \right) L^\ell. \quad (0.5)$$

The commutator of $P_{2n+1}$ and $L$ then reads[2]

$$[P_{2n+1}, L] = 2f_{n+1,x}, \quad (0.6)$$

using the recursion (0.3). Introducing a deformation (time) parameter[3] $t_n \in \mathbb{R}$, $n \in \mathbb{N}_0$ into $u$, the *KdV hierarchy* of nonlinear evolution equations is then defined by imposing the *Lax commutator relations*

$$\frac{d}{dt_n}L - [P_{2n+1}, L] = 0, \quad (0.7)$$

for each $n \in \mathbb{N}_0$. By (0.6), the latter are equivalent to the collection of evolution equations[4]

$$\mathrm{KdV}_n(u) = u_{t_n} - 2f_{n+1,x}(u) = 0, \quad n \in \mathbb{N}_0. \quad (0.8)$$

Explicitly,

$$\begin{aligned}
\mathrm{KdV}_0(u) &= u_{t_0} - u_x = 0, \\
\mathrm{KdV}_1(u) &= u_{t_1} + \tfrac{1}{4}u_{xxx} - \tfrac{3}{2}uu_x - c_1u_x = 0, \\
\mathrm{KdV}_2(u) &= u_{t_2} - \tfrac{1}{16}u_{xxxxx} + \tfrac{5}{8}uu_{xxx} + \tfrac{5}{4}u_xu_{xx} - \tfrac{15}{8}u^2u_x \\
&\quad + c_1\left(\tfrac{1}{4}u_{xxx} - \tfrac{3}{2}uu_x\right) - c_2u_x = 0, \quad \text{etc.},
\end{aligned}$$

[1] All details of the following construction are to be found in Chapter 1.

[2] The quantities $P_{2n+1}$ and $\{f_\ell\}_{\ell=0,\dots,n}$ are constructed in such a manner that all differential operators in the commutator (0.6) vanish.

[3] Here we follow Hirota's notation and introduce a separate time variable $t_n$ for the $n$th level in the KdV hierarchy.

[4] In a slight abuse of notation, we will occasionally stress the functional dependence of $f_\ell$ on $u$, writing $f_\ell(u)$.

represent the first few equations of the time-dependent KdV hierarchy. For $n = 1$ and $c_1 = 0$, we obtain *the* KdV equation (0.2). Introducing the polynomials ($z \in \mathbb{C}$),

$$F_n(z) = \sum_{\ell=0}^{n} f_{n-\ell} z^{\ell}, \tag{0.9}$$

$$G_{n-1}(z) = -F_{n,x}(z)/2, \tag{0.10}$$

$$H_{n+1}(z) = (z-u)F_n(z) + (1/2)F_{n,xx}(z), \tag{0.11}$$

one can alternatively introduce the KdV hierarchy as follows. One defines a pair of $2 \times 2$ matrices $(U(z), V_{n+1}(z))$ depending polynomially on $z$ by

$$U(z) = \begin{pmatrix} 0 & 1 \\ -z+u & 0 \end{pmatrix}, \tag{0.12}$$

$$V_{n+1}(z) = \begin{pmatrix} G_{n-1}(z) & F_n(z) \\ -H_{n+1}(z) & -G_{n-1}(z) \end{pmatrix}, \tag{0.13}$$

and then postulates the *zero-curvature equation*[1]

$$U_{t_n} - V_{n+1,x} + [U, V_{n+1}] = 0. \tag{0.14}$$

One easily verifies that both the Lax approach (0.8) as well as the zero-curvature approach (0.14) reduce to the basic equation

$$u_{t_n} + (1/2)F_{n,xxx} - 2(u-z)F_{n,x} - u_x F_n = 0. \tag{0.15}$$

Each one of (0.8), (0.14), and (0.15) defines the KdV hierarchy by varying $n \in \mathbb{N}_0$.

The strategy is as follows: We temporarily assume existence of a solution $u$ and derive several of its properties. In particular, we show that $u$ satisfies a trace formula (cf. (0.37) in the stationary case and (0.54) in the time-dependent case) expressed in terms of certain Dirichlet data that satisfy the so-called Dubrovin equations (cf. (0.38) in the stationary case and (0.55) in the time-dependent case), a first-order system of ordinary differential equations that can be shown at least locally to possess solutions. Furthermore, we deduce explicit formulas for the solution $u$, the so-called Its–Matveev formulas (cf. (0.40) in the stationary case and (0.57) in the time-dependent case).

The Lax and zero-curvature equations (0.7) and (0.14) imply a most remarkable isospectral deformation of $L$, as will be discussed later in this introduction. At this

[1] Equations $\Psi_x = U\Psi$, $\Psi_{t_n} = V_{n+1}\Psi$ and their compatibility condition (0.14), $U_{t_n} - V_{n+1,x} + [U, V_{n+1}] = 0$ permit a geometrical interpretation as follows: $U$ and $V_{n+1}$ may be considered local connection coefficients in the trivial vector bundle $\mathbb{R}^2 \times \mathbb{C}^2$ with space-time $\mathbb{R}^2$ the base and $\Psi$ taking values in the fiber $\mathbb{C}^2$. The compatibility equation (0.14) then shows that the $(U, V_{n+1})$-connection has zero-curvature, and hence (0.14) is called a zero-curvature representation of a nonlinear evolution equation.

point, however, we interrupt our time-dependent KdV considerations for a while and take a closer look at the special stationary KdV equations defined by

$$u_{t_n} = 0, \quad n \in \mathbb{N}_0. \tag{0.16}$$

By (0.6)–(0.8) and (0.14), (0.15), the condition (0.16) is then equivalent to each one of the following collection of equations, with $n$ ranging in $\mathbb{N}_0$, which then defines the *stationary KdV hierarchy*,

$$[P_{2n+1}, L] = 0, \tag{0.17}$$

$$f_{n+1,x} = 0, \tag{0.18}$$

$$-V_{n+1,x} + [U, V_{n+1}] = 0, \tag{0.19}$$

$$(1/2)F_{n,xxx} - 2(u-z)F_{n,x} - u_x F_n = 0. \tag{0.20}$$

To set the stationary KdV hierarchy apart from the general time-dependent one, we will denote it by

$$\text{s-KdV}_n(u) = -2f_{n+1,x}(u) = 0, \quad n \in \mathbb{N}_0.$$

Explicitly, the first few equations of the stationary KdV hierarchy then read as follows

$$\begin{aligned}
\text{s-KdV}_0(u) &= -u_x = 0, \\
\text{s-KdV}_1(u) &= \tfrac{1}{4}u_{xxx} - \tfrac{3}{2}uu_x - c_1 u_x = 0, \\
\text{s-KdV}_2(u) &= -\tfrac{1}{16}u_{xxxxx} + \tfrac{5}{8}uu_{xxx} + \tfrac{5}{4}u_x u_{xx} - \tfrac{15}{8}u^2 u_x \\
&\quad + c_1\left(\tfrac{1}{4}u_{xxx} - \tfrac{3}{2}uu_x\right) - c_2 u_x = 0, \quad \text{etc.}
\end{aligned}$$

The class of *algebro-geometric* KdV potentials, by definition, equals the set of solutions $u$ of the stationary KdV hierarchy. In the following analysis we fix the value of $n$ in (0.17)–(0.20), and hence we now turn to the investigation of algebro-geometric solutions $u$ of the $n$th equation within the stationary KdV hierarchy. Equation (0.17) is of special interest because, by a 1923 result of Burchnall and Chaundy, commuting differential expressions (due to a common eigenfunction to be discussed below, cf. (0.33), (0.34)) give rise to an algebraic relationship between the two differential expressions. Similarly, (0.19) permits the important conclusion that

$$\partial_x \det(yI_2 - iV_{n+1}(z,x)) = 0 \tag{0.21}$$

and hence

$$\begin{aligned}
\det(yI_2 - iV_{n+1}(z,x)) &= y^2 - \det(V_{n+1}(z,x)) \\
&= y^2 + G_{n-1}(z,x)^2 - F_n(z,x)H_{n+1}(z,x) = y^2 - R_{2n+1}(z)
\end{aligned} \tag{0.22}$$

for some $x$-independent monic polynomial $R_{2n+1}$, which we write as

$$R_{2n+1}(z) = \prod_{m=0}^{2n}(z - E_m) \text{ for some } \{E_m\}_{m=0,\dots,2n} \subset \mathbb{C}.$$

In particular, the combination

$$F_n(z,x)H_{n+1}(z,x) - G_{n-1}(z,x)^2 = R_{2n+1}(z) \tag{0.23}$$

is $x$-independent. Moreover, (0.20) can easily be integrated to yield

$$(1/2)F_{n,xx}F_n - (1/4)F_{n,x}^2 - (u-z)F_n^2 = R_{2n+1} \tag{0.24}$$

with precisely the same integration constant $R_{2n+1}(z)$ as in (0.22). In fact, by (0.10) and (0.11), equations (0.23) and (0.24) are simply identical. Incidentally, the algebraic relationship between $L$ and $P_{2n+1}$ alluded to in connection with the vanishing of their commutator in (0.17) can be made precise as follows: Restricting $P_{2n+1}$ to the (algebraic) kernel $\ker(L-z)$ of $L-z$, one computes, using (0.5) and (0.24),

$$\begin{aligned}\left(P_{2n+1}\big|_{\ker(L-z)}\right)^2 &= -\left(\frac{1}{2}F_{n,xx}F_n - \frac{1}{4}F_{n,x}^2 - (u-z)F_n^2\right)\Bigg|_{\ker(L-z)} \\ &= -R_{2n+1}(L)\big|_{\ker(L-z)}.\end{aligned}$$

Thus, one concludes that $P_{2n+1}^2$ and $-R_{2n+1}(L)$ coincide on $\ker(L-z)$, and since $z \in \mathbb{C}$ is arbitrary, one infers that

$$P_{2n+1}^2 + R_{2n+1}(L) = 0 \tag{0.25}$$

holds once again with the same polynomial $R_{2n+1}$. The characteristic equation of $iV_{n+1}$ (cf. (0.22)) and (0.25) naturally lead one to the introduction of the *hyperelliptic curve* $\mathcal{K}_n$ of (arithmetic) genus $n \in \mathbb{N}_0$ (possibly with a singular affine part) defined by

$$\mathcal{K}_n : \mathcal{F}_n(z,y) = y^2 - R_{2n+1}(z) = 0, \quad R_{2n+1}(z) = \prod_{m=0}^{2n}(z - E_m). \tag{0.26}$$

We compactify the curve by adding the point $P_\infty$ (still denoting it by $\mathcal{K}_n$ for simplicity) and note that points $P$ on the curve are denoted by $P = (z,y) \in \mathcal{K}_n \setminus \{P_\infty\}$, where $y(\cdot)$ is a meromorphic function on $\mathcal{K}_n$ satisfying[1] $y^2 - R_{2n+1}(z) = 0$. For simplicity, we will assume in the following that the (affine part of the) curve $\mathcal{K}_n$ is nonsingular, that is, the zeros $E_m$ of $R_{2n+1}$ are all simple. Remaining within the stationary framework a bit longer, one can now introduce the fundamental meromorphic function $\phi$ on $\mathcal{K}_n$ alluded to earlier as follows,

$$\phi(P,x) = \frac{iy - G_{n-1,x}(z,x)}{F_n(z,x)} \tag{0.27}$$

$$= \frac{-H_{n+1}(z,x)}{iy + G_{n-1,x}(z,x)}, \quad P = (z,y) \in \mathcal{K}_n. \tag{0.28}$$

[1] For more details, refer to Appendix B and Chapter 1.

Equality of the two expressions (0.27) and (0.28) is an immediate consequence of the identity (0.23) and the fact $y^2 = R_{2n+1}(z)$. A comparison with (0.19) then readily reveals that $\phi$ satisfies the Riccati-type equation

$$\phi_x + \phi^2 = u - z. \tag{0.29}$$

The next step is crucial. It concerns the zeros and poles of $\phi$ and hence involves the zeros of $F_n(\,\cdot\,, x)$ and $H_{n+1}(\,\cdot\,, x)$. Isolating the latter by introducing the factorizations

$$F_n(z,x) = \prod_{j=1}^{n}(z - \mu_j(x)), \quad H_{n+1}(z,x) = \prod_{\ell=0}^{n}(z - \nu_\ell(x)),$$

one can use the zeros of $F_n$ and $H_{n+1}$ to define the following points $\hat{\mu}_j(x)$, $\hat{\nu}_\ell(x)$ on $\mathcal{K}_n$,

$$\hat{\mu}_j(x) = (\mu_j(x), iG_{n-1,x}(\mu_j(x), x)), \quad j = 1, \ldots, n, \tag{0.30}$$

$$\hat{\nu}_\ell(x) = (\nu_\ell(x), -iG_{n-1,x}(\nu_\ell(x), x)), \quad \ell = 0, \ldots, n. \tag{0.31}$$

The motivation for this choice stems from $y^2 = R_{2n+1}(z)$ by (0.22), the identity (0.23) (which combines to $F_n H_{n+1} - G_{n-1}^2 = y^2$), and a comparison of (0.27) and (0.28). Given (0.27)–(0.31), one obtains for the divisor $(\phi(\,\cdot\,, x))$ of the meromorphic function $\phi$

$$(\phi(\,\cdot\,, x)) = \mathcal{D}_{\hat{\nu}_0(x)\underline{\hat{\nu}}(x)} - \mathcal{D}_{P_\infty\underline{\hat{\mu}}(x)}. \tag{0.32}$$

Here we abbreviated $\underline{\hat{\mu}} = \{\hat{\mu}_1, \ldots, \hat{\mu}_n\}, \underline{\hat{\nu}} = \{\hat{\nu}_1, \ldots, \hat{\nu}_n\} \in \operatorname{Sym}^n(\mathcal{K}_n)$, with $\operatorname{Sym}^n(\mathcal{K}_n)$ the $n$th symmetric product of $\mathcal{K}_n$, and used our conventions[1] (A.43), (A.47), and (A.48) to denote positive divisors of degree $n$ and $n+1$ on $\mathcal{K}_n$. Given $\phi(\,\cdot\,, x)$, one defines the *stationary Baker–Akhiezer vector* $\Psi(\,\cdot\,, x, x_0)$ on $\mathcal{K}_n \setminus \{P_\infty\}$ by

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \psi_1(P, x, x_0) = \exp\left(\int_{x_0}^{x} dx'\, \phi(P, x')\right), \quad \psi_2 = \psi_{1,x}.$$

In particular, this implies

$$\phi = \psi_2/\psi_1$$

and the following normalization[2] of $\psi_1$, $\psi_1(P, x_0, x_0) = 1$, $P \in \mathcal{K}_n \setminus \{P_\infty\}$. The Riccati-type equation (0.29) satisfied by $\phi$ then shows that the Baker–Akhiezer

[1] $\mathcal{D}_Q(P) = m$ if $P$ occurs $m$ times in $\{Q_1, \ldots, Q_n\}$ and zero otherwise, $\underline{Q} = \{Q_1, \ldots, Q_n\} \in \operatorname{Sym}^n(\mathcal{K}_n)$. Similarly, $\mathcal{D}_{Q_0\underline{Q}} = \mathcal{D}_{Q_0} + \mathcal{D}_{\underline{Q}}$, $\mathcal{D}_{\underline{Q}} = \mathcal{D}_{Q_1} + \cdots + \mathcal{D}_{Q_n}$, $Q_0 \in \mathcal{K}_n$, and $\mathcal{D}_Q(P) = 1$ for $P = Q$ and zero otherwise.

[2] This normalization is less innocent than it might appear at first sight. It implies that $\mathcal{D}_{\underline{\hat{\mu}}(x)}$ and $\mathcal{D}_{\underline{\hat{\mu}}(x_0)}$ are the divisors of zeros and poles of $\psi_1(\,\cdot\,, x, x_0)$ on $\mathcal{K}_n \setminus \{P_\infty\}$.

function $\psi_1$ is the common formal eigenfunction of the commuting pair of Lax differential expressions $L$ and $P_{2n+1}$,

$$L\psi_1(P) = z\psi_1(P), \tag{0.33}$$

$$P_{n+1}\psi_1(P) = iy\psi_1(P), \quad P = (z, y), \tag{0.34}$$

and at the same time the Baker–Akhiezer vector $\Psi$ satisfies the zero-curvature equations,

$$\Psi_x(P) = U(z)\Psi(P), \tag{0.35}$$

$$iy\Psi(P) = V_{n+1}(z)\Psi(P), \quad P = (z, y). \tag{0.36}$$

Moreover, one easily verifies that away from the (finite) branch points $(E_m, 0)$, $m = 0, \dots, 2n$, of the two-sheeted Riemann surface $\mathcal{K}_n$, the two branches of $\psi_1$ constitute a fundamental system of solutions of (0.33) and similarly, the two branches of $\Psi$ yield a fundamental system of solutions of (0.35). Since $\psi_1(\,\cdot\,, x, x_0)$ vanishes at $\hat{\mu}_j(x)$, $j = 1, \dots, n$ and $\psi_2(\,\cdot\,, x, x_0) = \psi_{1,x}(\,\cdot\,, x, x_0)$ vanishes at $\hat{\nu}_\ell(x)$, $\ell = 0, \dots, n$, we may call $\{\hat{\mu}_j(x)\}_{j=1,\dots,n}$ and $\{\hat{\nu}_\ell(x)\}_{\ell=0,\dots,n}$ the *Dirichlet* and *Neumann data* of $L$ at the point $x \in \mathbb{R}$, respectively.

Now the stationary formalism is almost complete; we only need to relate the solution $u$ of the $n$th stationary KdV equation and $\mathcal{K}_n$-associated data. This can be accomplished in several ways. We describe two of them next.

First we relate $u$ and the zeros $\mu_j$ of $F_n$. This is easily done by comparing the coefficients of the power $z^{2n}$ in (0.24) and results in the *trace formula*,

$$u = \sum_{m=0}^{2n} E_m - 2\sum_{j=1}^{n} \mu_j. \tag{0.37}$$

Next we will indicate how to reconstruct (at least locally) $u$ from Dirichlet data at just one fixed point $x_0$. Combining the definition (0.30) of $\hat{\mu}_j$ and that of $G_{n-1}$ in (0.10) yields, after a comparison with the $x$-derivative of $F_n(z, x) = \prod_{k=1}^{n}(z - \mu_k(x))$,

$$\begin{aligned} y(\hat{\mu}_j(x)) &= iG_{n-1}(\mu_j(x), x) = -(i/2)F_{n,x}(\mu_j(x), x) \\ &= (i/2)\mu_{j,x}(x)\prod_{\substack{k=1 \\ k\neq j}}^{n}(\mu_j(x) - \mu_k(x)), \quad j = 1, \dots, n. \end{aligned}$$

Hence, one arrives at the *Dubrovin equations* for $\hat{\mu}_j$, an autonomous first-order system of differential equations on $\mathcal{K}_n$,

$$\mu_{j,x} = -2iy(\hat{\mu}_j)\prod_{\substack{k=1 \\ k\neq j}}^{n}\left(\mu_j - \mu_k\right)^{-1}, \quad j = 1, \dots, n. \tag{0.38}$$