非线性动力学丛书

6

Singular Point Values, Center Problem and Bifurcations of Limit Cycles of Two Dimensional Differential Autonomous Systems

（二阶非线性系统的奇点量、中心问题与极限环分叉）

Liu Yirong, Li Jibin and Huang Wentao

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《非线性动力学丛书》序

真实的动力系统几乎都含有各种各样的非线性因素，诸如机械系统中的间隙、干摩擦，结构系统中的材料弹塑性、构件大变形，控制系统中的元器件饱和特性、变结构控制策略等等. 实践中，人们经常试图用线性模型来替代实际的非线性系统，以求方便地获得其动力学行为的某种逼近. 然而，被忽略的非线性因素常常会在分析和计算中引起无法接受的误差，使得线性逼近成为一场徒劳. 特别对于系统的长时间历程动力学问题，有时即使略去很微弱的非线性因素，也会在分析和计算中出现本质性的错误.

因此，人们很早就开始关注非线性系统的动力学问题. 早期研究可追溯到1673年 Huygens 对单摆大幅摆动非等时性的观察. 从 19 世纪末起, Poincaré、Lyapunov、Birkhoff、Andronov、Arnold 和 Smale 等数学家和力学家相继对非线性动力系统的理论进行了奠基性研究, Duffing、van der Pol、Lorenz、Ueda 等物理学家和工程师则在实验和数值模拟中获得了许多启示性发现. 他们的杰出贡献相辅相成，形成了分岔、混沌、分形的理论框架，使非线性动力学在 20 世纪 70 年代成为一门重要的前沿学科，并促进了非线性科学的形成和发展.

近 20 年来，非线性动力学在理论和应用两个方面均取得了很大进展. 这促使越来越多的学者基于非线性动力学观点来思考问题，采用非线性动力学理论和方法，对工程科学、生命科学、社会科学等领域中的非线性系统建立数学模型，预测其长期的动力学行为，揭示内在的规律性，提出改善系统品质的控制策略. 一系列成功的实践使人们认识到：许多过去无法解决的难题源于系统的非线性，而解决难题的关键在于对问题所呈现的分岔、混沌、分形、孤立子等复杂非线性动力学现象具有正确的认识和理解.

近年来，非线性动力学理论和方法正从低维向高维乃至无穷维发展. 伴随着计算机代数、数值模拟和图形技术的进步，非线性动力学所处理的问题规模和难度不断提高. 已逐步接近一些实际系统. 在工程科学界，以往研究人员对于非线性问题绕道而行的现象正在发生变化. 人们不仅力求深入分析非线性对系统动力学的影响，使系统和产品的动态设计、加工、运行与控制满足日益提高的运行速度和精度需求；而且开始探索利用分岔、混沌等非线性现象造福人类.

在这样的背景下, 有必要组织在工程科学、生命科学、社会科学等领域中从事非线性动力学研究的学者撰写一套非线性动力学丛书, 着重介绍近几年来非线性动力学理论和方法在上述领域的一些研究进展, 特别是我国学者的研究成果, 为从事非线性动力学理论及应用研究的人员, 包括硕士研究生和博士研究生等, 提供最新的理论、方法及应用范例. 在科学出版社的大力支持下, 组织了这套《非线性动力学丛书》.

本套丛书在选题和内容上有别于郝柏林先生主编的《非线性科学丛书》(上海教育出版社出版), 它更加侧重于对工程科学、生命科学、社会科学等领域中的非线性动力学问题进行建模、理论分析、计算和实验. 与国外的同类丛书相比, 它更具有整体的出版思想, 每分册阐述一个主题, 互不重复等特点. 丛书的选题主要来自我国学者在国家自然科学基金等资助下取得的研究成果, 有些研究成果已被国内外学者广泛引用或应用于工程和社会实践, 还有一些选题取自作者多年的教学成果.

希望作者、读者、丛书编委会和科学出版社共同努力, 使这套丛书取得成功.

胡海岩

2001 年 8 月

Preface

The qualitative theory and stability theory of differential equations, created by Poincaré and Lyapunov at the end of the 19th century had major developments as two branches of the theory of dynamical systems during the 20th century. As a part of the basic theory of nonlinear science, it is one of the very active areas in the new millennium.

This book presents in an elementary way the recent significant developments in the qualitative theory of planar dynamical systems. The subjects are covered as follows: the studies of center and isochronous center problems, multiple Hopf bifurcations and local and global bifurcations of the equivariant planar vector fields which concern with Hilbert's 16th problem.

We are interested in the study of planar vector fields, because they occur very often in applications. Indeed, such equations appear in modelling chemical reactions, population dynamics, traveling wave systems of nonlinear evolution equations in mathematical physics and in many other areas of applied mathematics and mechanics. In the other hand, the study of planar vector fields has itself theoretical signification. We would like to cite Canada's mathematician Dana Schlomiuk's words to explain this fact: "Planar polynomial vector fields and more generally, algebraic differential equations over the projective space are interesting objects of study for their own sake. Indeed, due to their analytic, algebraic and geometric nature they form a fertile soil for intertwining diverse methods, and success in finding solutions to problems in this area depends very much on the capacity we have to blend the diverse aspects into a unified whole."

We emphasize that for the problems of the planar vector fields, many sophisticated tools and theories have been built and still being developed, whose field of application goes far beyond the initial areas. In this book, we only state some important progress in the above directions which have attracted our study interest.

In order to clearly understand the content in this book for young readers, and to save space in the following chapters, we shall describe in more detail the subjects which are written in this book and give brief survey of the historic literature.

I. Center-focus problem

We consider planar vector fields and their associated differential equations:

$$\frac{dx}{dt} = X(x, y), \quad \frac{dy}{dt} = Y(x, y), \tag{E}$$

where $X(x,y), Y(x,y)$ are analytic functions or polynomials with real coefficients. If X, Y are polynomials, we call degree of a system (E), the number $n = \max(\deg(X), \deg(Y))$. Without loss of generality, we assume that $X(0,0) = Y(0,0) = 0$, i.e., the origin $O(0,0)$ is a singular point of (E) and the linearization at the origin of (E) has purely imaginary eigenvalues.

The origin $O(0,0)$ is a *center* of (E) if there exists a neighborhood U of the origin such that every point in U other than $O(0,0)$ is nonsingular and the orbit passing through the point is closed. In 1885, Poincaré posed the following problem.

The problem of the center

Find necessary and sufficient conditions for a planar polynomial differential system (E) of degree m to possess a center.

This problem is still unsolved for systems of degree greater than two.

Poincaré considered the above problem. He gives an infinite set of necessary and sufficient conditions for such system to have a center at the origin. In his memoir on the stability of motion, Lyapunov studies systems of differential equations in n variables. When applied to the case $n = 2$, his results also give a infinite set of necessary and sufficient conditions for system (E) with X, Y polynomials to have a center (actually, Lyapunov's result is more general since it is for the case where X and Y are analytic functions). In searching for sufficient conditions for a center, both Poincaré and Lyapunov's work involve the idea of trying to find a constant of the motion $F(x,y)$ for (E) in a neighborhood U of the origin, where

$$F(x,y) = \sum_{k=2}^{\infty} F_k(x,y), \tag{1}$$

F_k is a homogeneous polynomial of order k and F_2 is a positive definite quadratic form. If F is constant on all solution curve $(x(t), y(t))$ in U, we say that F is a first integral on U of system (E). If there exists such a F which is nonconstant on any open subset of U, we say that system (E) is integrable on U.

Poincaré and Lyapunov proved the following theorem.

Poincaré-Lyapunov Theorem The origin of the polynomial (or analytic) system (E) is a center if and only if in an open neighborhood U of the origin, (E) has a nonconstant first integral which is analytic.

Thus, we can construct a power series (1) such that

$$\left.\frac{dF}{dt}\right|_{(E)} = V_1(x^2+y^2)^2 + \cdots + V_k(x^2+y^2)^{k+1} + \cdots \tag{2}$$

with $V_1, V_2, \cdots, V_k, \cdots$ constants. The first non-zero V_i give the asymptotic stability or instability of the origin according to its negative or positive sign. Indeed, stopping the series at F_k, we obtain a polynomial which is a Lyapunov function for the system (E). The $V_i's$ are called the *Lyapunov constants.* Some people also use the term *focal values* for them. In fact, Andronov et al defined the focal values by the formula $\alpha_i = \dfrac{d^{(i)}(0)}{i!}$ which is the ith derivative of the function $d(\rho_0) = P(\rho_0) - \rho_0$, where P is the Poincaré return map. The first non-zero focal value of Andronov corresponding to an odd number $i = 2n+1$. It had been proved that the first non-zero Lyapunov constant V_n differs only by a positive constant factor from the first non-zero focal value, which is $d^{(2n+1)}(0)$. Hence, the identification in the terminology is natural.

In terms of the $V_i's$, the conditions for a center of the origin become $V_k = 0$, for all $k = 1, 2, 3, \cdots$. Now $V_1, V_2, \cdots, V_k, \cdots$ are polynomial with rational coefficients in the coefficients of $X(x,y)$ and $Y(x,y)$. Theoretically, by using Hilbert's basis theorem, the ideal generated by these polynomials has a finite basis $B_1, B_2, \cdots, B_m$. Hence, we have a finite set of necessary and sufficient conditions for a center, i.e., $B_i = 0$ for $i = 1, 2, \cdots, M$. To calculate this basis, we reduce each V_k modulo $\ll V_1, V_2, \cdots, V_{k-1} \gg$, the ideal generated by $V_1, V_2, \cdots, V_{k-1}$. The elements of the basis thus obtained are called the Lyapunov quantities or the focal quantities. The origin is said to be a k-order fine focus (or a focus of multiplicity k) of (E) if the fist $k-1$ Lyapunov quantities are 0 but the k-order one is not.

The above statement tell us that the solution of the center-focus for a particular system, the procedure is as follows: compute several Lyapunov constants and when we get one significant constant that is zero, try to prove that the system obtained indeed has a center. Unfortunately, the described method has the following questions.

(1) How can we be sure that you have computed enough Lyapunov constants?

(2) How do we prove that some system candidate to have a center actually has a center?

(3) Do you know the general construction of Lyapunov constants in order to get general shortened expressions for Lyapunov constants $V_1, V_2, \cdots$.

In Chapter 1 and Chapter 2 we devote to give possible answer for these questions.

In addition, we shall consider the following two problems.

Problem of center-focus at infinite singular point

A real planar polynomial vector field V can be compactified on the sphere as follows: Consider the x, y plane as being the plane $Z = 1$ in the space $\mathcal{R}^3$ with coordinates X, Y, Z. The center projection of the vector field V on the sphere of radius one yields a diffeomorphic vector field on the upper hemisphere and also another vector field on the lower hemisphere. There exists an analytic vector filed $p(V)$ on the whole sphere such that its restriction on the upper hemisphere has the same phase curves as the one constructed above from the polynomial vector field. The projection of the closed northern hemisphere H^+ of $\mathcal{S}^2$ on $Z = 0$ under $(X, Y, Z) \to (X, Y)$ is called the *Poincaré disc.* A singular point q of $p(V)$ is called a *infinite* (or finite) singular point if $q \in \mathcal{S}^1$ (or $q \in \mathcal{S}^2$ $\mathcal{S}^1$). The vector field $p(V)$ restricted to the upper hemisphere completed with the equator is called *Poincaré compactification of a polynomial vector field.*

For a infinite singular point, there exists the problem of the characterization of center for concrete families of planar polynomial (or analytic) systems. In chapter 2, we shall introduce our some research results.

Problem of center-focus at a high-order singular point

The center-focus problem for a degenerate singular point is essentially difficult problems. There is only a few results on this direction before 2000 year. This book shall give some basic results in Chapter 2.

II. Small-amplitude limit cycles created by multiple Hopf bifurcations

So called *Hopf bifurcation*, it means that a differential system exhibits the phenomenon that the appearance of periodic solution (or limit cycle in plane) branching off from an equilibrium point of the system when certain changes of the parameters occur. Hopf's original work on this subject appeared in 1942, in which the author considered higher dimensional (greater than two) systems. Before 1940s, Andronov and his co-workers had done the pioneering work for planar dynamical systems. Bautin showed that for planar quadratic systems at most three small-amplitude limit cycles can bifurcate out of one equilibrium point. By the work of Andronov et al, it is well known that the bifurcation of several limit cycles from a fine focus is directly related with the stability of the focus. The sign of the first nonvanishing Lyapunov constant determines the stability of the focus. Furthermore, the number of the leading $V_i's(i = 1, 2, \cdots)$ which vanish simultaneously is the number of limit cycles which may bifurcate from the focus. This is the reason why the investigation of the bifurcation of limit cycles deal with the computation of Lyaponov constants.

The appearance of more than one limit cycles from one equilibrium point is called *multiple Hopf bifurcation.* How these small-amplitude limit cycles can be generated? The idea is to start with a system (E) for which the origin is a kth fine focus, then to make a sequence of perturbations of the coefficients of $X(x,y)$ and $Y(x,y)$ each of which reverses the stability of the origin, thereby causing a limit cycle to bifurcate.

In Chapter 3 and Chapter 4 the readers shall see a lot of examples of systems having multiple Hopf bifurcation.

III. Local and non-local bifurcations of Z_q-equivariant perturbed planar Hamiltonian vector fields

The second part of Hilbert's 16th problem deals with the maximum number $H(n)$ and relative positions of limit cycles of a polynomial system

$$\frac{dx}{dt} = P_n(x,y), \qquad \frac{dy}{dt} = Q_n(x,y) \tag{E_n}$$

of degree n, i.e., $\max(\deg P, \deg Q) = n$. Hilbert conjectured that the number of limit cycles of (E_n) is bounded by a number depending only on the degree n of the vector fields.

Let χ_N be the space of planar vector fields $X = \left(P_n = \sum\limits_{i+j=0}^{n} a_{ij}x^iy^j, Q_n = \sum\limits_{i+j=0}^{n} b_{ij}x^iy^j\right)$ with the coefficients $(a_{ij}, b_{ij}) \in B \subset R^N$, for $0 \leqslant i+j \leqslant n, N = (n+1)(n+2)$. The standard procedure in the study of polynomial vector fields is to consider their behavior at infinity by extension to the Poincaré sphere. Thus, we can see (E_n) as an analytic N-parameter family of differential equations on S^2 with the compact base B. Then, the second part of Hilbert's 16th problem may be splited into three parts:

Problem A Prove the finiteness of the number of limit cycles for any concrete system $X \in \chi_N$(given a particular choice for coefficients of (E_n)) i.e,

$$\sharp\{\mathrm{L.C.} \;\; of \;\; (\mathrm{E}_n)\} < \infty.$$

Problem B Prove for every n the existence of a uniformly bounded upper bound for the number of limit cycles on the set B as the function of the parameters, i.e.,

$$\forall n, \;\; \forall \; (a_{ij}, b_{ij}) \in B, \exists H(n) \;\; such \; that \;\; \sharp\{\mathrm{L.C.} \;\; of \;\; (\mathrm{E}_n)\} \leqslant H(n)$$

and find an upper estimate for $H(n)$.

Problem C For every n and known $K = H(n)$, find all possible configurations (or schemes) of limit cycles for every number $K, K-i, i = 1, 2, \cdots, K-1$ respectively.

Hence, the second part of Hilbert's 16th problem consists of problem A∼problem C.

The problem A for polynomial and analytic differential equations are already solved by Écalle (1992) and Ilyashenko (1991) independently. Of course, as S. Small stated that "These two papers have yet to be thoroughly digested by mathematical community".

Up to now, there is no approach to the solution of the problem B, even for $n = 2$, which seem to be very complicated. But there exists a similar problem, which seems to be a little bit easier. It is the weakened Hilbert's 16th problem proposed by Arnold (1977):

"Let H be a real polynomial of degree n and let P be a real polynomial of degree m in the variables (x, y). How many real zeroes can the function

$$I(h) = \int\int_{H \leqslant h} P dx dy$$

have ? "

The question is why zeroes of the Abelian integrals $I(h)$ is concerned with the second part of Hilbert's 16th problem ?

Let $H(x, y)$ be a real polynomial of degree n, and let $P(x, y)$ and $Q(x, y)$ be real polynomials of degree m. We consider a perturbed Hamiltonian system in the form

$$\frac{dx}{dt} = \frac{\partial H}{\partial y} + \varepsilon P(x, y, \lambda), \quad \frac{dy}{dt} = -\frac{\partial H}{\partial x} + \varepsilon Q(x, y, \lambda), \tag{E_H}$$

in which we assume that $0 < \varepsilon \ll 1$ and the level curves

$$H(x, y) = h$$

of the Hamiltonian system $(E_H)_{\varepsilon=0}$ contain at least a family Γ_h of closed orbits for $h \in (h_1, h_2)$.

Consider the Abelian integrals

$$I(h) = \int_{\Gamma_h} (P(x, y)dy - Q(x, y)dx) = \int\int_{H \leqslant h} \left(\frac{\partial P(x, y)}{\partial x} + \frac{\partial Q(x, y)}{\partial y} \right) dx dy.$$

Poincaré-Pontrjagin-Andronov Theorem on the global center bifurcation

The following statements hold.

(i) If $I(h^*) = 0$ and $I'(h^*) \neq 0$, then there exists a hyperbolic limit cycle L_{h^*} of system (6.1) such that $L_{h^*} \to \Gamma_{h^*}$ as $\varepsilon \to 0$; and conversely, if there exists a hyperbolic limit cycle L_{h^*} of system (E_H) such that $L_{h^*} \to \Gamma_{h^*}$ as $\varepsilon \to 0$, then $I(h^*) = 0$, where $h^* \in (h_1, h_2)$.

(ii) If $I(h^*) = I'(h^*) = I''(h^*) = \cdots = I^{(k-1)}(h^*) = 0$, and $I^{(k)}(h^*) \neq 0$, then (E_H) has at most k limit cycles for ε sufficiently small in the vicinity of Γ_{h^*}.

(iii) The total number of isolated zeroes of the Abelian integral (taking into account their multiplicity) is an upper bound for the number of limit cycles of system (E_H) that bifurcate from the periodic orbits of a period annulus of Hamiltonian system $(\mathrm{E}_H)_{\varepsilon=0}$.

This theorem tells us that the weakened Hilbert's 16th problem posed by Arnold (1977) is closely related to the problem of determining an upper bound $N(n, m) = N(n, m, H, P, Q)$ for the number of limit cycles in a period annulus for the Hamiltonian system of degree $n - 1$ under the perturbations of degree m, i.e., of determining the cyclicity on a period annulus. Since the problem is concerned with the number of limit cycles that occur in systems which are close to integrable ones (only a class of subsystems of all polynomial systems). So that it is called the weakened Hilbert's 16th problem.

A closed orbit Γ_{h^*} satisfying the above theorem (i) is called a generating cycle.

To obtain Poincaré-Pontrjagin-Andronov Theorem, the problem for investigating the bifurcated limit cycles is based on the Poincaré return mapping. It is reduced to counting the number of zeroes of the displacement function

$$d(h, \varepsilon) = \varepsilon M_1(h) + \varepsilon^2 M_2(h) + \cdots + \varepsilon^k M_k(h) + \cdots,$$

where $d(h, \varepsilon)$ is defined on a section to the flow, which is parameterized by the Hamiltonian value h. $I(h)$ just is equal to $M_1(h)$. The function $M_k(h)$ is called k order Melnikov function. If $I(h) = M_1(h) \equiv 0$, we need to estimate the number of zeroes of higher order Melnikov functions. The zeroes of the first nonvanishing Melnikov function $M_k(h)$ determine the limit cycles in (E_H) emerging from periodic orbits of the Hamiltonian system $(\mathrm{E}_H)_\varepsilon$.

In Chapter 5 we discuss a class of particular polynomial vector fields—Z_q-equivariant perturbed planar Hamiltonian vector fields, by using Poincaré-Pontrjagin-Andronov Theorem and Melnikov's result. The aim is to get some information for the second part of Hilbert's 16th problem.

IV. Isochronous center problem and periodic map

Suppose that system (E) has a center in the origin $(0,0)$. Then, there is a family of periodic orbits of (E) enclosing the origin. The largest neighborhood of the center entirely covered by periodic orbits is called a *period annulus* of the center. If the period of the orbits is constant for all periodic orbits lying in the period annulus of the origin, then the center $(0,0)$ is called an *isochronous center*. It has been proved that the isochronous center can exist if the period annulus of the center is unbounded.

If the origin is not an isochronous center, for a point $(\xi, 0)$ in a small neighborhood of the origin $(0,0)$, we define $P(\xi)$ to be the minimum period of the periodic orbit passing through $(\xi, 0)$. The study for the period function $\xi \to P(\xi)$ is also very interesting problem, since monotonicity of the period function is a non-degeneracy condition for the bifurcation of subharmonic solutions of periodically forced integrable systems.

The history of the work on period functions goes back at least to 1673 when C. Huygens observed that the pendulum clock has a monotone period function and therefore oscillates with a shorter period when the energy is decreased, i.e., as the clock spring unwinds. He hope to design a clock with isochronous oscillations in order to have a more accurate clock to be used in the navigation of ships. His solution, the cycloidal pendulum, is perhaps the first example of nonlinear isochronous center.

In the last three decades of the 20th century, a considerable number of papers of the study for isochronous centers and period maps has been published. But, for a given polynomial vector field of the degree is more than two, the characterization of isochronous center is still a very difficult, challenging and unsolved problem.

In Chapter 6 we introduce some new method to treat these problems.

In Chapter 7 we consider a class of nonanalytic systems which is called "quasi-analytic systems". We will completely solve its center and isochronous center problems as well as the bifurcation of limit cycles.

Finally, in Chapter 8, as an example, for a class of Z_2-symmetric cubic systems, we give the complete answer for the center problem.

We would like to cite the following words written by Anna Schlomiuk in 2004 as the finale of this preface: "Planar polynomial vector fields are dynamical systems but to perceive them uniquely from this angle is limiting, missing part of their essence and hampering development of their theory. Indeed, as dynamical systems they are very special systems and the prevalent generic viewpoint pushes them on the side. This may explain in part why Hilbert's 16th problem as well as other problems are still unsolved even in their simple case, the quadratic one. But, Poincaré's work shows that he regarded these systems as interesting object of study from several viewpoints, and his appreciation of the work of Darboux which he qualifies as "admirable" emphasizes this point. This area is rich with problems, very hard, it is true, but exactly for this reason an open mind and a free flow of ideas is necessary. It is to be hoped that in the future there will be a better understanding of this area which lies at a crossroads of dynamical systems, algebra, geometry and where algebraic and geometric problems go hand in hand with those of dynamical systems."

The book is intended for graduate students, post-doctors and researchers in dynamical systems. For all engineers who are interested the theory of dynamical systems, it is also a reasonable reference. It requires a minimum background of a one-year course on nonlinear differential equations.

The publication of this book is supported by the research foundation of the Center for Dynamical Systems and Nonlinear Science Studies given by Zhejiang Normal University. The work described in this book is supported by the grants from the National Natural Science Foundation of China (No.10231020 and No.10771196) and the National Natural Science Foundation of Yunnan Province and Science Foundation of Guilin University of Electronic Technology, partly.

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Li Jibin
Spring, 2007

Contents

Chapter 1

Focal Values, Saddle Values and Singular Point Values

In this chapter, we consider a class of real planar autonomous differential systems, for which the functions of the right hand are analytic in a neighborhood of the origin and the origin is a focus or a center. We shall introduce the elementary theory to solve the center problem.

1.1 Successor Functions and Properties of Focal Values

By making a linear change of the space coordinates and a rescaling of the time variable if necessary, a planar differential system can be written as

$$\begin{aligned}\frac{dx}{dt} &= \delta x - y + \sum_{k=2}^{\infty} X_k(x,y) = X(x,y),\\ \frac{dy}{dt} &= x + \delta y + \sum_{k=2}^{\infty} Y_k(x,y) = Y(x,y),\end{aligned} \tag{1.1.1}$$

where $X(x,y)$, $Y(x,y)$ are analytic in a sufficiently small neighborhood of the origin, and

$$\begin{aligned}X_k(x,y) &= \sum_{\alpha+\beta=k} A_{\alpha\beta} x^{\alpha} y^{\beta},\\ Y_k(x,y) &= \sum_{\alpha+\beta=k} B_{\alpha\beta} x^{\alpha} y^{\beta}.\end{aligned} \tag{1.1.2}$$

It is well known that the origin of system (1.1.1) is a simple focus when $\delta \neq 0$ and it is either a weak focus or a center when $\delta = 0$. The problem of determining whether a non-degenerate critical point (it has purely imaginary eigenvalues) is a center or a weak focus is called the center-focus problem (or simply, center problem). This is one of the most important topics in the qualitative theory of planar dynamical systems. Poincaré (1891~1897), Lyapunov