

Graduate Texts in Mathematics

H.H. Schaefer

With M.P. Wolff

Topological Vector Spaces

Second Edition

拓扑向量空间

第2版

Springer

世界图书出版公司
www.wpcbj.com.cn

H.H. Schaefer
With M.P. Wolff

Topological Vector Spaces

Second Edition



Springer

图书在版编目 (CIP) 数据

拓扑向量空间: 第2版: 英文/ (英) 舍费尔 (Schaefer, H. H.) 著. —北京: 世界图书出版公司北京公司, 2009. 4

书名原文: Topological Vector Spaces 2nd ed.

ISBN 978-7-5100-0446-9

I. 拓… II. 舍… III. 向量拓扑—拓扑空间—高等学校—教材—英文 IV. 0189. 11

中国版本图书馆 CIP 数据核字 (2009) 第 048107 号

书 名: Topological Vector Spaces 2nd ed.

作 者: H. H. Schaefer

中译名: 拓扑向量空间 第2版

责任编辑: 高蓉 刘慧

出 版 者: 世界图书出版公司北京公司

印 刷 者: 三河国英印务有限公司

发 行: 世界图书出版公司北京公司 (北京朝内大街 137 号 100010)

联系电话: 010-64021602, 010-64015659

电子信箱: kjb@wpcbj.com.cn

开 本: 24 开

印 张: 15.5

版 次: 2009 年 04 月

版权登记: 图字: 01-2009-0687

书 号: 978-7-5100-0446-9/0 · 661

定 价: 48.00 元

世界图书出版公司北京公司已获得 Springer 授权在中国大陆独家重印发行

H.H. Schaefer
M.P. Wolff
Eberhard-Karls-Universität Tübingen
Mathematisches Institut
Auf der Morgenstelle 10
Tübingen, 72076
Germany

Editorial Board

S. Axler
Mathematics Department
San Francisco State
University
San Francisco, CA 94132
USA

F. W. Gehring
Mathematics Department
East Hall
University of Michigan
Ann Arbor, MI 48109
USA

K. A. Ribet
Mathematics
Department
University of
California at
Berkeley
Berkeley, CA 94720-3840
USA

Mathematics Subject Classification (1991): 46-01, 46Axx, 46Lxx

Library of Congress Cataloging-in-Publication Data

Schaefer, Helmut H.

Topological vector spaces. — 2nd ed. / Helmut H. Schaefer in
assistance with M. Wolff.

p. cm. — (Graduate texts in mathematics ; 3)

Includes bibliographical references and indexes.

ISBN 0-387-98726-6 (alk. paper)

I. Linear topological spaces. I. Wolff, Manfred, 1939–

II. Title. III. Series.

QA322.S28 1999

514'.32—dc21

98-53842

First edition © 1966 by H. H. Schaefer. Published by the Macmillan Company, New York.

© 1999 Springer-Verlag New York, Inc.

All rights reserved. This work may not be translated or copied in whole or in part without the written permission of the publisher (Springer-Verlag New York, Inc., 175 Fifth Avenue, New York, NY 10010, USA), except for brief excerpts in connection with reviews or scholarly analysis. Use in connection with any form of information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed is forbidden.

The use of general descriptive names, trade names, trademarks, etc., in this publication, even if the former are not especially identified, is not to be taken as a sign that such names, as understood by the Trade Marks and Merchandise Marks Act, may accordingly be used freely by anyone.

This reprint has been authorized by Springer-Verlag (Berlin/Heidelberg/New York) for sale in the Mainland China only and not for export therefrom.

ISBN 0-387-98726-6 Springer-Verlag New York Berlin Heidelberg SPIN 10707604

Graduate Texts in Mathematics 3

Editorial Board

S. Axler F.W. Gehring K.A. Ribet

Springer

New York

Berlin

Heidelberg

Barcelona

Hong Kong

London

Milan

Paris

Singapore

Tokyo

Graduate Texts in Mathematics

- 1 TAKEUTI/ZARING. Introduction to Axiomatic Set Theory. 2nd ed.
- 2 OXToby. Measure and Category. 2nd ed.
- 3 SCHAEFER. Topological Vector Spaces. 2nd ed.
- 4 HILTON/STAMMBACH. A Course in Homological Algebra. 2nd ed.
- 5 MAC LANE. Categories for the Working Mathematician. 2nd ed.
- 6 HUGHES/PIPER. Projective Planes.
- 7 SERRE. A Course in Arithmetic.
- 8 TAKEUTI/ZARING. Axiomatic Set Theory.
- 9 HUMPHREYS. Introduction to Lie Algebras and Representation Theory.
- 10 COHEN. A Course in Simple Homotopy Theory.
- 11 CONWAY. Functions of One Complex Variable I. 2nd ed.
- 12 BEALS. Advanced Mathematical Analysis.
- 13 ANDERSON/FULLER. Rings and Categories of Modules. 2nd ed.
- 14 GOLUBITSKY/GUILLEMIN. Stable Mappings and Their Singularities.
- 15 BERBERIAN. Lectures in Functional Analysis and Operator Theory.
- 16 WINTER. The Structure of Fields.
- 17 ROSENBLATT. Random Processes. 2nd ed.
- 18 HALMOS. Measure Theory.
- 19 HALMOS. A Hilbert Space Problem Book. 2nd ed.
- 20 HUSEMOLLER. Fibre Bundles. 3rd ed.
- 21 HUMPHREYS. Linear Algebraic Groups.
- 22 BARNES/MACK. An Algebraic Introduction to Mathematical Logic.
- 23 GREUB. Linear Algebra. 4th ed.
- 24 HOLMES. Geometric Functional Analysis and Its Applications.
- 25 HEWITT/STROMBERG. Real and Abstract Analysis.
- 26 MANES. Algebraic Theories.
- 27 KELLEY. General Topology.
- 28 ZARISKI/SAMUEL. Commutative Algebra. Vol. I.
- 29 ZARISKI/SAMUEL. Commutative Algebra. Vol. II.
- 30 JACOBSON. Lectures in Abstract Algebra I. Basic Concepts.
- 31 JACOBSON. Lectures in Abstract Algebra II. Linear Algebra.
- 32 JACOBSON. Lectures in Abstract Algebra III. Theory of Fields and Galois Theory.
- 33 HIRSCH. Differential Topology.
- 34 SPITZER. Principles of Random Walk. 2nd ed.
- 35 ALEXANDER/WERMER. Several Complex Variables and Banach Algebras. 3rd ed.
- 36 KELLEY/NAMIOKA et al. Linear Topological Spaces.
- 37 MONK. Mathematical Logic.
- 38 GRAUERT/FRITZSCHE. Several Complex Variables.
- 39 ARVESON. An Invitation to C^* -Algebras.
- 40 KEMENY/SNELL/KNAPP. Denumerable Markov Chains. 2nd ed.
- 41 APOSTOL. Modular Functions and Dirichlet Series in Number Theory. 2nd ed.
- 42 SERRE. Linear Representations of Finite Groups.
- 43 GILLMAN/JERISON. Rings of Continuous Functions.
- 44 KENDIG. Elementary Algebraic Geometry.
- 45 LOÈVE. Probability Theory I. 4th ed.
- 46 LOÈVE. Probability Theory II. 4th ed.
- 47 MOISE. Geometric Topology in Dimensions 2 and 3.
- 48 SACHS/WU. General Relativity for Mathematicians.
- 49 GRUENBERG/WEIR. Linear Geometry. 2nd ed.
- 50 EDWARDS. Fermat's Last Theorem.
- 51 KLINGENBERG. A Course in Differential Geometry.
- 52 HARTSHORNE. Algebraic Geometry.
- 53 MANIN. A Course in Mathematical Logic.
- 54 GRAVER/WATKINS. Combinatorics with Emphasis on the Theory of Graphs.
- 55 BROWN/PEARCY. Introduction to Operator Theory I: Elements of Functional Analysis.
- 56 MASSEY. Algebraic Topology: An Introduction.
- 57 CROWELL/FOX. Introduction to Knot Theory.
- 58 KOBLITZ. p -adic Numbers, p -adic Analysis, and Zeta-Functions. 2nd ed.
- 59 LANG. Cyclotomic Fields.
- 60 ARNOLD. Mathematical Methods in Classical Mechanics. 2nd ed.
- 61 WHITEHEAD. Elements of Homotopy Theory.

(continued after index)

PREFACE TO THE SECOND EDITION

As the first edition of this book has been well received through five printings over a period of more than thirty years, we have decided to leave the material of the first edition essentially unchanged – barring a few necessary updates. On the other hand, it appeared worthwhile to extend the existing text by adding a reasonably informative introduction to C^* - and W^* -algebras. The theory of these algebras seems to be of increasing importance in mathematics and theoretical physics, while being intimately related to topological vector spaces and their orderings—the prime concern of this text.

The authors wish to thank J. Schweizer for a careful reading of Chapter VI, and the publisher for their care and assistance.

Tübingen, Germany
Spring 1999

H. H. Schaefer
M. P. Wolff

Preface

The present book is intended to be a systematic text on topological vector spaces and presupposes familiarity with the elements of general topology and linear algebra. The author has found it unnecessary to rederive these results, since they are equally basic for many other areas of mathematics, and every beginning graduate student is likely to have made their acquaintance. Similarly, the elementary facts on Hilbert and Banach spaces are widely known and are not discussed in detail in this book, which is mainly addressed to those readers who have attained and wish to get beyond the introductory level.

The book has its origin in courses given by the author at Washington State University, the University of Michigan, and the University of Tübingen in the years 1958–1963. At that time there existed no reasonably complete text on topological vector spaces in English, and there seemed to be a genuine need for a book on this subject. This situation changed in 1963 with the appearance of the book by Kelley, Namioka *et al.* [1] which, through its many elegant proofs, has had some influence on the final draft of this manuscript. Yet the two books appear to be sufficiently different in spirit and subject matter to justify the publication of this manuscript; in particular, the present book includes a discussion of topological tensor products, nuclear spaces, ordered topological vector spaces, and an appendix on positive operators. The author is also glad to acknowledge the strong influence of Bourbaki, whose monograph [7], [8] was (before the publication of Köthe [5]) the only modern treatment of topological vector spaces in printed form.

A few words should be said about the organization of the book. There is a preliminary chapter called “Prerequisites,” which is a survey aimed at clarifying the terminology to be used and at recalling basic definitions and facts to the reader’s mind. Each of the five following chapters, as well as the Appendix, is divided into sections. In each section, propositions are marked $u.v$, where u is the section number, v the proposition number within the

section. Propositions of special importance are additionally marked "Theorem." Cross references within the chapter are (u.v), outside the chapter (r, u.v), where r (roman numeral) is the number of the chapter referred to. Each chapter is preceded by an introduction and followed by exercises. These "Exercises" (a total of 142) are devoted to further results and supplements, in particular, to examples and counter-examples. They are not meant to be worked out one after the other, but every reader should take notice of them because of their informative value. We have refrained from marking some of them as difficult, because the difficulty of a given problem is a highly subjective matter. However, hints have been given where it seemed appropriate, and occasional references indicate literature that may be needed, or at least helpful. The bibliography, far from being complete, contains (with few exceptions) only those items that are referred to in the text.

I wish to thank A. Pietsch for reading the entire manuscript, and A. L. Peressini and B. J. Walsh for reading parts of it. My special thanks are extended to H. Lotz for a close examination of the entire manuscript, and for many valuable discussions. Finally, I am indebted to H. Lotz and A. L. Peressini for reading the proofs, and to the publisher for their care and cooperation.

H. H. S.

Tübingen, Germany
December, 1964

To my wife

Table of Contents

<i>Preface to the Second Edition</i>	v
<i>Preface</i>	vi

Prerequisites

A. <i>Sets and Order</i>	1
B. <i>General Topology</i>	4
C. <i>Linear Algebra</i>	9

I. TOPOLOGICAL VECTOR SPACES

Introduction	12
1 Vector Space Topologies	12
2 Product Spaces, Subspaces, Direct Sums, Quotient Spaces	19
3 Topological Vector Spaces of Finite Dimension	21
4 Linear Manifolds and Hyperplanes	24
5 Bounded Sets	25
6 Metrizability	28
7 Complexification	31
Exercises	33

II. LOCALLY CONVEX TOPOLOGICAL VECTOR SPACES

Introduction	36
1 Convex Sets and Semi-Norms	37
2 Normed and Normable Spaces	40
3 The Hahn-Banach Theorem	45

TABLE OF CONTENTS

ix

4	Locally Convex Spaces	47
5	Projective Topologies	51
6	Inductive Topologies	54
7	Barreled Spaces	60
8	Bornological Spaces	61
9	Separation of Convex Sets	63
10	Compact Convex Sets	66
	Exercises	68

III. LINEAR MAPPINGS

	Introduction	73
1	Continuous Linear Maps and Topological Homomorphisms	74
2	Banach's Homomorphism Theorem	76
3	Spaces of Linear Mappings	79
4	Equicontinuity. The Principle of Uniform Boundedness and the Banach-Steinhaus Theorem	82
5	Bilinear Mappings	87
6	Topological Tensor Products	92
7	Nuclear Mappings and Spaces	97
8	Examples of Nuclear Spaces	106
9	The Approximation Property. Compact Maps	108
	Exercises	115

IV. DUALITY

	Introduction	122
1	Dual Systems and Weak Topologies	123
2	Elementary Properties of Adjoint Maps	128
3	Locally Convex Topologies Consistent with a Given Duality. The Mackey-Arens Theorem	130
4	Duality of Projective and Inductive Topologies	133
5	Strong Dual of a Locally Convex Space. Bidual. Reflexive Spaces	140
6	Dual Characterization of Completeness. Metrizable Spaces. Theorems of Grothendieck, Banach-Dieudonné, and Krein-Šmulian	147

7	Adjoins of Closed Linear Mappings	155
8	The General Open Mapping and Closed Graph Theorems	161
9	Tensor Products and Nuclear Spaces	167
10	Nuclear Spaces and Absolute Summability	176
11	Weak Compactness. Theorems of Eberlein and Krein	185
	Exercises	190

V. ORDER STRUCTURES

	Introduction	203
1	Ordered Vector Spaces over the Real Field	204
2	Ordered Vector Spaces over the Complex Field	214
3	Duality of Convex Cones	215
4	Ordered Topological Vector Spaces	222
5	Positive Linear Forms and Mappings	225
6	The Order Topology	230
7	Topological Vector Lattices	234
8	Continuous Functions on a Compact Space. Theorems of Stone-Weierstrass and Kakutani	242
	Exercises	250

VI. C^* - AND W^* -ALGEBRAS

	Introduction	258
1	Preliminaries	259
2	C^* -Algebras. The Gelfand Theorem	260
3	Order Structure of a C^* -Algebra	267
4	Positive Linear Forms. Representations	270
5	Projections and Extreme Points	274
6	W^* -Algebras	277
7	Von Neumann Algebras. Kaplansky's Density Theorem	287
8	Projections and Types of W^* -Algebras	292
	Exercises	299

Appendix. SPECTRAL PROPERTIES OF POSITIVE OPERATORS

	Introduction	306
1	Elementary Properties of the Resolvent	307
2	Pringsheim's Theorem and Its Consequences	309

TABLE OF CONTENTS

xi

3 The Peripheral Point Spectrum

316

Index of Symbols

325

Bibliography

330

Index

339

PREREQUISITES

A formal prerequisite for an intelligent reading of this book is familiarity with the most basic facts of set theory, general topology, and linear algebra. The purpose of this preliminary section is not to establish these results but to clarify terminology and notation, and to give the reader a survey of the material that will be assumed as known in the sequel. In addition, some of the literature is pointed out where adequate information and further references can be found.

Throughout the book, statements intended to represent definitions are distinguished by setting the term being defined in bold face characters.

A. SETS AND ORDER

1. *Sets and Subsets.* Let X, Y be sets. We use the standard notations $x \in X$ for “ x is an element of X ”, $X \subset Y$ (or $Y \supset X$) for “ X is a subset of Y ”, $X = Y$ for “ $X \subset Y$ and $Y \subset X$ ”. If (p) is a proposition in terms of given relations on X , the subset of all $x \in X$ for which (p) is true is denoted by $\{x \in X: (p)x\}$ or, if no confusion is likely to occur, by $\{x: (p)x\}$. $x \notin X$ means “ x is not an element of X ”. The **complement** of X relative to Y is the set $\{x \in Y: x \notin X\}$, and denoted by $Y \sim X$. The empty set is denoted by \emptyset and considered to be a finite set; the set (**singleton**) containing the single element x is denoted by $\{x\}$. If $(p_1), (p_2)$ are propositions in terms of given relations on X , $(p_1) \Rightarrow (p_2)$ means “ (p_1) implies (p_2) ”, and $(p_1) \Leftrightarrow (p_2)$ means “ (p_1) is equivalent with (p_2) ”. The set of all subsets of X is denoted by $\mathfrak{P}(X)$.

2. *Mappings.* A mapping f of X into Y is denoted by $f: X \rightarrow Y$ or by $x \rightarrow f(x)$. X is called the **domain** of f , the image of X under f , the **range** of f ; the **graph** of f is the subset $G_f = \{(x, f(x)): x \in X\}$ of $X \times Y$. The mapping of the set $\mathfrak{P}(X)$ of all subsets of X into $\mathfrak{P}(Y)$ that is associated with f , is also denoted by f ; that is, for any $A \subset X$ we write $f(A)$ to denote the set

$\{f(x) : x \in A\} \subset Y$. The associated map of $\mathfrak{P}(Y)$ into $\mathfrak{P}(X)$ is denoted by f^{-1} ; thus for any $B \subset Y$, $f^{-1}(B) = \{x \in X : f(x) \in B\}$. If $B = \{b\}$, we write $f^{-1}(b)$ in place of the clumsy (but more precise) notation $f^{-1}(\{b\})$. If $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are maps, the composition map $x \rightarrow g(f(x))$ is denoted by $g \circ f$.

A map $f: X \rightarrow Y$ is **biunivocal (one-to-one, injective)** if $f(x_1) = f(x_2)$ implies $x_1 = x_2$; it is **onto Y (surjective)** if $f(X) = Y$. A map f which is both injective and surjective is called **bijective** (or a **bijection**).

If $f: X \rightarrow Y$ is a map and $A \subset X$, the map $g: A \rightarrow Y$ defined by $g(x) = f(x)$ whenever $x \in A$ is called the **restriction** of f to A and frequently denoted by f_A . Conversely, f is called an **extension** of g (to X with values in Y).

3. *Families.* If A is a *non-empty* set and X is a set, a mapping $\alpha \rightarrow x(\alpha)$ of A into X is also called a **family** in X ; in practice, the term family is used for mappings whose domain A enters only in terms of its set theoretic properties (i.e., cardinality and possibly order). One writes, in this case, x_α for $x(\alpha)$ and denotes the family by $\{x_\alpha : \alpha \in A\}$. Thus every non-empty set X can be viewed as the family (identity map) $x \rightarrow x(x \in X)$; but it is important to notice that if $\{x_\alpha : \alpha \in A\}$ is a family in X , then $\alpha \neq \beta$ does not imply $x_\alpha \neq x_\beta$. A **sequence** is a family $\{x_n : n \in N\}$, $N = \{1, 2, 3, \dots\}$ denoting the set of natural numbers. If confusion with singletons is unlikely and the domain (index set) A is clear from the context, a family will sometimes be denoted by $\{x_\alpha\}$ (in particular, a sequence by $\{x_n\}$).

4. *Set Operations.* Let $\{X_\alpha : \alpha \in A\}$ be a family of sets. For the union of this family, we use the notations $\bigcup \{X_\alpha : \alpha \in A\}$, $\bigcup_{\alpha \in A} X_\alpha$, or briefly $\bigcup_\alpha X_\alpha$ if the index set A is clear from the context. If $\{X_n : n \in N\}$ is a sequence of sets we also write $\bigcup_1^\infty X_n$, and if $\{X_1, \dots, X_k\}$ is a finite family of sets we write $\bigcup_1^k X_n$ or $X_1 \cup X_2 \cup \dots \cup X_k$. Similar notations are used for intersections and Cartesian products, with \bigcup replaced by \bigcap and \prod respectively. If $\{X_\alpha : \alpha \in A\}$ is a family such that $X_\alpha = X$ for all $\alpha \in A$, the product $\prod_\alpha X_\alpha$ is also denoted by X^A .

If R is an equivalence relation (i.e., a reflexive, symmetric, transitive binary relation) on the set X , the set of equivalence classes (the **quotient set**) by R is denoted by X/R . The map $x \rightarrow \hat{x}$ (also denoted by $x \rightarrow [x]$) which orders to each x its equivalence class \hat{x} (or $[x]$), is called the **canonical** (or **quotient**) map of X onto X/R .

5. *Orderings.* An **ordering (order structure, order)** on a set X is a binary relation R , usually denoted by \leq , on X which is reflexive, transitive, and anti-symmetric ($x \leq y$ and $y \leq x$ imply $x = y$). The set X endowed with an order \leq is called an **ordered set**. We write $y \geq x$ to mean $x \leq y$, and $x < y$ to mean $x \leq y$ but $x \neq y$ (similarly for $x > y$). If R_1 and R_2 are orderings of X , we say that R_1 is **finer** than R_2 (or that R_2 is **coarser** than R_1) if $x(R_1)y$ implies $x(R_2)y$. (Note that this defines an ordering on the set of all orderings of X .)

Let (X, \leq) be an ordered set. A subset A of X is **majorized** if there exists $a_0 \in X$ such that $a \leq a_0$ whenever $a \in A$; a_0 is a **majorant** (**upper bound**) of A . Dually, A is **minorized** by a_0 if $a_0 \leq a$ whenever $a \in A$; then a_0 is a **minorant** (**lower bound**) of A . A subset A which is both majorized and minorized, is called **order bounded**. If A is majorized and there exists a majorant a_0 such that $a_0 \leq b$ for any majorant b of A , then a_0 is unique and called the **supremum** (**least upper bound**) of A ; the notation is $a_0 = \sup A$. In a dual fashion, one defines the **infimum** (**greatest lower bound**) of A , to be denoted by $\inf A$. For each pair $(x, y) \in X \times X$, the supremum and infimum of the set $\{x, y\}$ (whenver they exist) are denoted by $\sup(x, y)$ and $\inf(x, y)$ respectively. (X, \leq) is called a **lattice** if for each pair (x, y) , $\sup(x, y)$ and $\inf(x, y)$ exist, and (X, \leq) is called a **complete lattice** if $\sup A$ and $\inf A$ exist for every non-empty subset $A \subset X$. (In general we avoid this latter terminology because of the possible confusion with uniform completeness.) (X, \leq) is **totally ordered** if for each pair (x, y) , at least one of the relations $x \leq y$ and $y \leq x$ is true. An element $x \in X$ is **maximal** if $x \leq y$ implies $x = y$.

Let (X, \leq) be a *non-empty* ordered set. X is called **directed under \leq** (briefly, **directed** (\leq)) if every subset $\{x, y\}$ (hence each finite subset) possesses an upper bound. If $x_0 \in X$, the subset $\{x \in X : x_0 \leq x\}$ is called a **section of X** (more precisely, the section of X **generated by x_0**). A family $\{y_\alpha : \alpha \in A\}$ is **directed** if A is a directed set; the **sections** of a directed family are the sub-families $\{y_\alpha : \alpha_0 \leq \alpha\}$, for any $\alpha_0 \in A$.

Finally, an ordered set X is **inductively ordered** if each totally ordered subset possesses an upper bound. In each inductively ordered set, there exist maximal elements (Zorn's lemma). In most applications of Zorn's lemma, the set in question is a family of subsets of a set S , ordered by set theoretical inclusion \subset .

6. **Filters.** Let X be a set. A set \mathfrak{F} of subsets of X is called a **filter on X** if it satisfies the following axioms:

- (1) $\mathfrak{F} \neq \emptyset$ and $\emptyset \notin \mathfrak{F}$.
- (2) $F \in \mathfrak{F}$ and $F \subset G \subset X$ implies $G \in \mathfrak{F}$.
- (3) $F \in \mathfrak{F}$ and $G \in \mathfrak{F}$ implies $F \cap G \in \mathfrak{F}$.

A set \mathfrak{B} of subsets of X is a **filter base** if (1') $\mathfrak{B} \neq \emptyset$ and $\emptyset \notin \mathfrak{B}$, and (2') if $B_1 \in \mathfrak{B}$ and $B_2 \in \mathfrak{B}$ there exists $B_3 \in \mathfrak{B}$ such that $B_3 \subset B_1 \cap B_2$. Every filter base \mathfrak{B} generates a unique filter \mathfrak{F} on X such that $F \in \mathfrak{F}$ if and only if $B \subset F$ for at least one $B \in \mathfrak{B}$; \mathfrak{B} is called a **base of the filter \mathfrak{F}** . The set of all filters on a non-empty set X is inductively ordered by the relation $\mathfrak{F}_1 \subset \mathfrak{F}_2$ (set theoretic inclusion of $\mathfrak{P}(X)$); $\mathfrak{F}_1 \subset \mathfrak{F}_2$ is expressed by saying that \mathfrak{F}_1 is **coarser** than \mathfrak{F}_2 , or that \mathfrak{F}_2 is **finer** than \mathfrak{F}_1 . Every filter on X which is maximal with respect to this ordering, is called an **ultrafilter** on X ; by Zorn's lemma, for each filter \mathfrak{F} on X there exists an ultrafilter finer than \mathfrak{F} : If $\{x_\alpha : \alpha \in A\}$ is a directed family in X , the ranges of the sections of this family form a filter base on X ; the corresponding filter is called the **section filter** of the family.