Graduate Texts in Mathematics

H.H. Schaefer

With M.P. Wolff

Topological Vector Spaces

Second Edition

拓扑向量空间

第2版

Springer

世界图书出版公司
www.wpcbj.com.cn

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图书在版编目（CIP）数据

拓扑向量空间：第2版：英文/（英）舍费尔（Schaefer，H. H.）著.—北京：世界图书出版公司北京公司，2009.4

书名原文：Topological Vector Spaces 2nd ed.

ISBN 978-7-5100-0446-9

Ⅰ.拓… Ⅱ.舍… Ⅲ.向量拓扑—拓扑空间—高等学校—教材—英文 Ⅳ.O189.11

中国版本图书馆CIP数据核字（2009）第048107号

书　　名：	Topological Vector Spaces 2nd ed.
作　　者：	H. H. Schaefer
中 译 名：	拓扑向量空间 第2版
责任编辑：	高蓉　刘慧
出 版 者：	世界图书出版公司北京公司
印 刷 者：	三河国英印务有限公司
发　　行：	世界图书出版公司北京公司（北京朝内大街137号　100010）
联系电话：	010-64021602，010-64015659
电子信箱：	kjb@wpcbj.com.cn
开　　本：	24开
印　　张：	15.5
版　　次：	2009年04月
版权登记：	图字：01-2009-0687
书　　号：	978-7-5100-0446-9/O·661　　**定　　价：** 48.00元

H.H. Schaefer
M.P. Wolff
Eberhard-Karls-Universitat Tübingen
Mathematisches Institut
Auf der Morgenstelle 10
Tübingen, 72076
Germany

Mathematics Subject Classification (1991): 46-01, 46Axx, 46Lxx

Library of Congress Cataloging-in-Publication Data
Schaefer, Helmut H.
Topological vector spaces. — 2nd ed. / Helmut H. Schaefer in assistance with M. Wolff.
p. cm. — (Graduate texts in mathematics ; 3)
Includes bibliographical references and indexes.
ISBN 0-387-98726-6 (alk. paper)
1. Linear topological spaces. I. Wolff, Manfred, 1939–
II. Title. III. Series.
QA322.S28 1999
514'.32—dc21 98-53842

ISBN 0-387-98726-6 Springer-Verlag New York Berlin Heidelberg SPIN 10707604

Graduate Texts in Mathematics 3

Springer
New York
Berlin
Heidelberg
Barcelona
Hong Kong
London
Milan
Paris
Singapore
Tokyo

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PREFACE TO THE SECOND EDITION

As the first edition of this book has been well received through five printings over a period of more than thirty years, we have decided to leave the material of the first edition essentially unchanged – barring a few necessary updates. On the other hand, it appeared worthwhile to extend the existing text by adding a reasonably informative introduction to C^*– and W^*–algebras. The theory of these algebras seems to be of increasing importance in mathematics and theoretical physics, while being intimately related to topological vector spaces and their orderings—the prime concern of this text.

The authors wish to thank J. Schweizer for a careful reading of Chapter VI, and the publisher for their care and assistance.

Tübingen, Germany
Spring 1999

H. H. Schaefer
M. P. Wolff

Preface

The present book is intended to be a systematic text on topological vector spaces and presupposes familiarity with the elements of general topology and linear algebra. The author has found it unnecessary to rederive these results, since they are equally basic for many other areas of mathematics, and every beginning graduate student is likely to have made their acquaintance. Similarly, the elementary facts on Hilbert and Banach spaces are widely known and are not discussed in detail in this book, which is mainly addressed to those readers who have attained and wish to get beyond the introductory level.

The book has its origin in courses given by the author at Washington State University, the University of Michigan, and the University of Tübingen in the years 1958–1963. At that time there existed no reasonably complete text on topological vector spaces in English, and there seemed to be a genuine need for a book on this subject. This situation changed in 1963 with the appearance of the book by Kelley, Namioka *et al.* [1] which, through its many elegant proofs, has had some influence on the final draft of this manuscript. Yet the two books appear to be sufficiently different in spirit and subject matter to justify the publication of this manuscript; in particular, the present book includes a discussion of topological tensor products, nuclear spaces, ordered topological vector spaces, and an appendix on positive operators. The author is also glad to acknowledge the strong influence of Bourbaki, whose monograph [7], [8] was (before the publication of Köthe [5]) the only modern treatment of topological vector spaces in printed form.

A few words should be said about the organization of the book. There is a preliminary chapter called "Prerequisites," which is a survey aimed at clarifying the terminology to be used and at recalling basic definitions and facts to the reader's mind. Each of the five following chapters, as well as the Appendix, is divided into sections. In each section, propositions are marked u.v, where u is the section number, v the proposition number within the section. Propositions of special importance are additionally marked "Theorem." Cross references within the chapter are (u.v), outside the chapter (r, u.v), where r (roman numeral) is the number of the chapter referred to. Each chapter is preceded by an introduction and followed by exercises. These "Exercises" (a total of 142) are devoted to further results and supplements, in particular, to examples and counter-examples. They are not meant to be worked out one after the other, but every reader should take notice of them because of their informative value. We have refrained from marking some of them as difficult, because the difficulty of a given problem is a highly subjective matter. However, hints have been given where it seemed appropriate, and occasional references indicate literature that may be needed, or at least helpful. The bibliography, far from being complete, contains (with few exceptions) only those items that are referred to in the text.

I wish to thank A. Pietsch for reading the entire manuscript, and A. L. Peressini and B. J. Walsh for reading parts of it. My special thanks are extended to H. Lotz for a close examination of the entire manuscript, and for many valuable discussions. Finally, I am indebted to H. Lotz and A. L. Peressini for reading the proofs, and to the publisher for their care and cooperation.

H. H. S.

Tübingen, Germany
December, 1964

To my wife

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PREREQUISITES

A formal prerequisite for an intelligent reading of this book is familiarity with the most basic facts of set theory, general topology, and linear algebra. The purpose of this preliminary section is not to establish these results but to clarify terminology and notation, and to give the reader a survey of the material that will be assumed as known in the sequel. In addition, some of the literature is pointed out where adequate information and further references can be found.

Throughout the book, statements intended to represent definitions are distinguished by setting the term being defined in bold face characters.

A. SETS AND ORDER

1. *Sets and Subsets.* Let X, Y be sets. We use the standard notations $x \in X$ for "x is an element of X", $X \subset Y$ (or $Y \supset X$) for "X is a subset of Y", $X = Y$ for "$X \subset Y$ and $Y \supset X$". If (p) is a proposition in terms of given relations on X, the subset of all $x \in X$ for which (p) is true is denoted by $\{x \in X: (p)x\}$ or, if no confusion is likely to occur, by $\{x: (p)x\}$. $x \notin X$ means "x is not an element of X". The **complement** of X relative to Y is the set $\{x \in Y: x \notin X\}$, and denoted by $Y \sim X$. The empty set is denoted by $\emptyset$ and considered to be a finite set; the set (**singleton**) containing the single element x is denoted by $\{x\}$. If (p_1), (p_2) are propositions in terms of given relations on X, $(p_1) \Rightarrow (p_2)$ means "(p_1) implies (p_2)", and $(p_1) \Leftrightarrow (p_2)$ means "(p_1) is equivalent with (p_2)". The set of all subsets of X is denoted by $\mathfrak{P}(X)$.

2. *Mappings.* A mapping f of X into Y is denoted by $f: X \to Y$ or by $x \to f(x)$. X is called the **domain** of f, the image of X under f, the **range** of f; the **graph** of f is the subset $G_f = \{(x, f(x)): x \in X\}$ of $X \times Y$. The mapping of the set $\mathfrak{P}(X)$ of all subsets of X into $\mathfrak{P}(Y)$ that is associated with f, is also denoted by f; that is, for any $A \subset X$ we write $f(A)$ to denote the set

$\{f(x): x \in A\} \subset Y$. The associated map of $\mathfrak{P}(Y)$ into $\mathfrak{P}(X)$ is denoted by f^{-1}; thus for any $B \subset Y$, $f^{-1}(B) = \{x \in X: f(x) \in B\}$. If $B = \{b\}$, we write $f^{-1}(b)$ in place of the clumsier (but more precise) notation $f^{-1}(\{b\})$. If $f: X \to Y$ and $g: Y \to Z$ are maps, the composition map $x \to g(f(x))$ is denoted by $g \circ f$.

A map $f: X \to Y$ is **biunivocal (one–to–one, injective)** if $f(x_1) = f(x_2)$ implies $x_1 = x_2$; it is **onto** Y **(surjective)** if $f(X) = Y$. A map f which is both injective and surjective is called **bijective** (or a **bijection**).

If $f: X \to Y$ is a map and $A \subset X$, the map $g: A \to Y$ defined by $g(x) = f(x)$ whenever $x \in A$ is called the **restriction** of f to A and frequently denoted by f_A. Conversely, f is called an **extension** of g (to X with values in Y).

3. *Families.* If A is a *non-empty* set and X is a set, a mapping $\alpha \to x(\alpha)$ of A into X is also called a **family** in X; in practice, the term family is used for mappings whose domain A enters only in terms of its set theoretic properties (i.e., cardinality and possibly order). One writes, in this case, x_α for $x(\alpha)$ and denotes the family by $\{x_\alpha: \alpha \in \mathrm{A}\}$. Thus every non-empty set X can be viewed as the family (identity map) $x \to x (x \in X)$; but it is important to notice that if $\{x_\alpha: \alpha \in \mathrm{A}\}$ is a family in X, then $\alpha \neq \beta$ does not imply $x_\alpha \neq x_\beta$. A **sequence** is a family $\{x_n: n \in N\}$, $N = \{1, 2, 3, \ldots\}$ denoting the set of natural numbers. If confusion with singletons is unlikely and the domain (index set) A is clear from the context, a family will sometimes be denoted by $\{x_\alpha\}$ (in particular, a sequence by $\{x_n\}$).

4. *Set Operations.* Let $\{X_\alpha: \alpha \in \mathrm{A}\}$ be a family of sets. For the union of this family, we use the notations $\bigcup\{X_\alpha: \alpha \in \mathrm{A}\}$, $\bigcup_{\alpha \in \mathrm{A}} X_\alpha$, or briefly $\bigcup_\alpha X_\alpha$ if the index set A is clear from the context. If $\{X_n: n \in N\}$ is a sequence of sets we also write $\bigcup_1^\infty X_n$, and if $\{X_1, \ldots, X_k\}$ is a finite family of sets we write $\bigcup_1^k X_n$ or $X_1 \cup X_2 \cup \ldots \cup X_k$. Similar notations are used for intersections and Cartesian products, with $\bigcup$ replaced by $\bigcap$ and $\prod$ respectively. If $\{X_\alpha: \alpha \in \mathrm{A}\}$ is a family such that $X_\alpha = X$ for all $\alpha \in \mathrm{A}$, the product $\prod_\alpha X_\alpha$ is also denoted by X^{A}.

If R is an equivalence relation (i.e., a reflexive, symmetric, transitive binary relation) on the set X, the set of equivalence classes (the **quotient set**) by R is denoted by X/R. The map $x \to \hat{x}$ (also denoted by $x \to [x]$) which orders to each x its equivalence class $\hat{x}$ (or $[x]$), is called the **canonical** (or **quotient**) map of X onto X/R.

5. *Orderings.* An **ordering (order structure, order)** on a set X is a binary relation R, usually denoted by $\leqq$, on X which is reflexive, transitive, and antisymmetric ($x \leqq y$ and $y \leqq x$ imply $x = y$). The set X endowed with an order $\leqq$ is called an **ordered** set. We write $y \geqq x$ to mean $x \leqq y$, and $x < y$ to mean $x \leqq y$ but $x \neq y$ (similarly for $x > y$). If R_1 and R_2 are orderings of X, we say that R_1 is **finer** than R_2 (or that R_2 is **coarser** than R_1) if $x(R_1)y$ implies $x(R_2)y$. (Note that this defines an ordering on the set of all orderings of X.)

Let $(X, \leqq)$ be an ordered set. A subset A of X is **majorized** if there exists $a_0 \in X$ such that $a \leqq a_0$ whenever $a \in A$; a_0 is a **majorant (upper bound)** of A. Dually, A is **minorized** by a_0 if $a_0 \leqq a$ whenever $a \in A$; then a_0 is a **minorant (lower bound)** of A. A subset A which is both majorized and minorized, is called **order bounded.** If A is majorized and there exists a majorant a_0 such that $a_0 \leqq b$ for any majorant b of A, then a_0 is unique and called the **supremum (least upper bound)** of A; the notation is $a_0 = \sup A$. In a dual fashion, one defines the **infimum (greatest lower bound)** of A, to be denoted by $\inf A$. For each pair $(x, y) \in X \times X$, the supremum and infimum of the set $\{x, y\}$ (whenever they exist) are denoted by $\sup(x, y)$ and $\inf(x, y)$ respectively. $(X, \leqq)$ is called a **lattice** if for each pair (x, y), $\sup(x, y)$ and $\inf(x, y)$ exist, and $(X, \leqq)$ is called a **complete lattice** if $\sup A$ and $\inf A$ exist for every non-empty subset $A \subset X$. (In general we avoid this latter terminology because of the possible confusion with uniform completeness.) $(X, \leqq)$ is **totally ordered** if for each pair (x, y), at least one of the relations $x \leqq y$ and $y \leqq x$ is true. An element $x \in X$ is **maximal** if $x \leqq y$ implies $x = y$.

Let $(X, \leqq)$ be a *non-empty* ordered set. X is called **directed** under $\leqq$ (briefly, directed $(\leqq)$) if every subset $\{x, y\}$ (hence each finite subset) possesses an upper bound. If $x_0 \in X$, the subset $\{x \in X : x_0 \leqq x\}$ is called a **section** of X (more precisely, the section of X **generated** by x_0). A family $\{y_\alpha : \alpha \in \mathrm{A}\}$ is **directed** if A is a directed set; the **sections** of a directed family are the subfamilies $\{y_\alpha : \alpha_0 \leqq \alpha\}$, for any $\alpha_0 \in \mathrm{A}$.

Finally, an ordered set X is **inductively ordered** if each totally ordered subset possesses an upper bound. In each inductively ordered set, there exist maximal elements (Zorn's lemma). In most applications of Zorn's lemma, the set in question is a family of subsets of a set S, ordered by set theoretical inclusion $\subset$.

6. *Filters*. Let X be a set. A set $\mathfrak{F}$ of subsets of X is called a **filter** on X if it satisfies the following axioms:

(1) *$\mathfrak{F} \neq \emptyset$ and $\emptyset \notin \mathfrak{F}$.*
(2) *$F \in \mathfrak{F}$ and $F \subset G \subset X$ implies $G \in \mathfrak{F}$.*
(3) *$F \in \mathfrak{F}$ and $G \in \mathfrak{F}$ implies $F \cap G \in \mathfrak{F}$.*

A set $\mathfrak{B}$ of subsets of X is a **filter base** if (1′) $\mathfrak{B} \neq \emptyset$ and $\emptyset \notin \mathfrak{B}$, and (2′) if $B_1 \in \mathfrak{B}$ and $B_2 \in \mathfrak{B}$ there exists $B_3 \in \mathfrak{B}$ such that $B_3 \subset B_1 \cap B_2$. Every filter base $\mathfrak{B}$ generates a unique filter $\mathfrak{F}$ on X such that $F \in \mathfrak{F}$ if and only if $B \subset F$ for at least one $B \in \mathfrak{B}$; $\mathfrak{B}$ is called a **base of the filter** $\mathfrak{F}$. The set of all filters on a non-empty set X is inductively ordered by the relation $\mathfrak{F}_1 \subset \mathfrak{F}_2$ (set theoretic inclusion of $\mathfrak{P}(X)$); $\mathfrak{F}_1 \subset \mathfrak{F}_2$ is expressed by saying that $\mathfrak{F}_1$ is **coarser** than $\mathfrak{F}_2$, or that $\mathfrak{F}_2$ is **finer** than $\mathfrak{F}_1$. Every filter on X which is maximal with respect to this ordering, is called an **ultrafilter** on X; by Zorn's lemma, for each filter $\mathfrak{F}$ on X there exists an ultrafilter finer than $\mathfrak{F}$. If $\{x_\alpha : \alpha \in \mathrm{A}\}$ is a directed family in X, the ranges of the sections of this family form a filter base on X; the corresponding filter is called the **section filter** of the family.