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Introduction to Theoretical Mechanics

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INTRODUCTION TO THEORETICAL MECHANICS

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The late F. K. Richtmyer was Consulting Editor of the series from its inception in 1929 to his death in 1939. Lee A. DuBridge was Consulting Editor from 1939 to 1946; and G. P. Harnwell from 1947 to 1954.

PREFACE

This book has evolved from a course on the subject which I have given at the University of Illinois for the last five or six years. The course is a two-semester one meeting three times a week and is required, in their junior or senior year, of all undergraduates in the curriculum of engineering physics. In addition, a good proportion of the incoming graduate students in physics customarily enroll in at least the second semester's work. This is especially true if the prior background of the student in this particular field of classical physics is considered to be inadequate. Since classical mechanics is a basis for most other advanced courses in physics, the student should acquire a good deal of facility with this fundamental subject before attempting to undertake more advanced work. In this book the only preparation expected of the student is that obtained from thorough courses in elementary physics and calculus.

The methods of vectors are employed rather extensively throughout the text. However, no previous preparation on the part of the student in this regard is assumed, and an introduction to the subject of vector analysis, adequate for the present text, is presented in Chap. 1. Similarly no prior knowledge of elementary differential equations is necessary, although it must be admitted that a subject such as mechanics necessarily involves a certain dependence on this important branch of mathematics. However, the organization of the book is such that those portions in which a slight knowledge of differential equations is of advantage occur in later chapters of the book. Even here the mathematical tools are developed where needed and frequent reference is made to Appendix 2 in which a very brief introduction to the subject of ordinary differential equations is presented.

The emphasis of the book is quite definitely toward the solution of problems and, although an effort has been made to include a few very easy ones in each chapter, some of the exercises are rather difficult and are calculated to require a good deal of ingenuity on the part of the student. Indeed, it is not to be expected that all students will be able to solve all of the 400-odd problems in the book without assistance. However, the serious student who rises to the challenge presented by some of the problems is certain to find himself amply rewarded. The problem emphasis is easily justified, since the surest way to cultivate an ability to do physical reasoning is to apply it. Very little indeed can be learned in a course in mechanics, or physics in general, by the majority of students if the course is purely of a lecture type in which the solving of problems plays but a minor role.

There are upwards of eighty rather carefully selected examples which are worked out in the text material of the chapters. In addition to amplifying the mathematical steps in these solutions, a serious attempt has been made to present extensive details of the physical reasoning involved in the problem. In studying these examples the student is strongly advised against simply reading through the solution given. Rather should he first read only the statement of the example and, following this, attempt to set up the problem himself. In this way the student's difficulty with certain aspects of the case will become much more apparent to him, details which might have passed unnoticed had he contented himself with merely reading through the solution given.

The book is arranged so that, with the possible exception of Chap. 4, topics occur in the order of increasing difficulty as to both mathematical maturity and physical insight required. For this reason such subjects as central field motion, accelerated coordinate systems, general rigid body motion, Lagrangian methods, vibrating systems having several degrees of freedom, and wave motion are relegated to the latter half of the book. Although the treatment is primarily intended for a two-semester course on mechanics, the arrangement and order of the topics presented is such that the first nine chapters suffice to meet the demands of most one-semester courses on the subject.

Certain features of the book reflect the trend of modern physics. In connection with oscillatory motion in one dimension brief mention is made of nonlinear systems, a topic of ever-increasing importance in modern technology. In the chapter dealing with theorems concerning systems of particles, the case of a body in which the mass is varying (witness the rocket) is considered, and the procedure for setting up the equation of motion for such a situation is described. More space than is usually customary is devoted to the subject of general rigid body rotations in space. This is in keeping with the present wide interest in the fields of magnetic resonance and microwave spectroscopy, with their obvious applications of this class of motions.

Generalized coordinate methods are not introduced until after rigid body motion is considered. I am strongly of the opinion that the student should be taken through the latter material once without the use of the more sophisticated procedures so as to acquire more of a feeling than might otherwise have been gained for the way in which the forces are acting and for the selection of suitable coordinate systems. Not to be overlooked, also, is the fact that fairly complicated nonholonomic problems frequently can be successfully attacked step by step with the less sophisticated methods (witness Prob. 12-19). Such cases often present difficulty when Lagrangian procedures involving the use of Lagrangian multipliers (not discussed in this text) are employed.

Vibrating systems of several degrees of freedom are considered in the light of normal coordinates. One system, the vibrating string, having a large number of degrees of freedom is treated both from the normal coordinate and traveling wave points of view.

In conclusion I wish to mention my great indebtedness to the Cambridge University Press, and to Ginn and Company for graciously granting me permission for the use of certain of the problems in the text. Those marked *C* in the text are taken from the Cambridge publications: "Statics," "Dynamics," and "Higher Mechanics," all by Lamb; "Dynamics of a Particle," and "Elementary Rigid Dynamics," both by Routh; Ramsey's "Dynamics"; and "Mechanics," by Love. Many of these problems are reprinted by these authors from former Cambridge examinations. A few problems, marked *J*, are taken from Jeans' "Theoretical Mechanics," published by Ginn and Company. The remainder of the problems are either of my own composition or are taken from former examination lists that have been used at the University of Illinois. Some also have been suggested by certain interested individuals.

Finally I wish to thank my colleagues for many helpful suggestions. I especially wish to thank Professor Ronald Geballe of the University of Washington, and Professors A. T. Nordsieck and C. P. Slichter of the University of Illinois for their valuable criticisms, suggestions, and comments.

ROBERT A. BECKER

CONTENTS

CHAPTER 1

FUNDAMENTAL PRINCIPLES

1-1. Introductory Remarks. In the present text many basic concepts will be assumed to be possessed intuitively by the student. Such geometrical terms as *position* and *length* have familiar connotations to all from everyday experience. To some extent these notions are rendered more precise by secondary school mathematics. The first lends meaning to the location of a point in space. The second provides a common basis for describing the distance along a prescribed path between two such points. If the element of *time* is added and if one inquires into the rate at which the distance is traversed, the discussion becomes *kinematical*. *Kinematics*, it may be said, is the geometry of motion. Typical kinematical quantities are *velocity* and *acceleration*. The addition of the concepts of *mass* and *force*, which are physical quantities, brings the considerations under the heading of *mechanics*. The concepts of mass and force are employed in any elementary text on physics and will be quantitatively defined later in the present volume in terms of Newton's laws.

In order to complete the list of elementary concepts, it is necessary to mention two terms which are frequently employed in the discussion of mechanical problems. These are the *particle* and the *body*. The first of these, an idealized construct which is convenient in many problems, is a mass which has no size associated with it. In brief it is a geometrical point which possesses mass. The body, on the other hand, in general possesses both mass and extent.

1-2. Coordinate Systems. A typical mechanical problem, as applied to a given system, is to determine the configuration of that system as a function of time. If it consists of a number of particles, the general problem will be to specify the positions and velocities of all the particles in terms of time as the independent variable. In practice, however, it may be sufficient to determine a much smaller amount of information. For example, it may be desired to know the way in which the velocity of one of the particles will vary as its position in space is varied.

In order to attack any problem of this nature, it is necessary first to select an appropriate coordinate system. We limit ourselves at this early stage to the familiar *rectangular* system of the type shown in Fig. 1-1.

The system $Oxyz$ has its yz plane in the plane of the paper, with y positive to the right and z positive upward. The x axis points out from the paper toward the reader and is positive in that direction.

In selecting suitable coordinates for a problem it is convenient to retain only the minimum number of distinct coordinates necessary to describe the motion completely. For example, if a particle is free to move in one plane, such as a table top, clearly only two coordinates will be necessary. We may choose the plane of the table to be the xy plane, as in Fig. 1-2. Suppose the particle is at point P at a given instant.

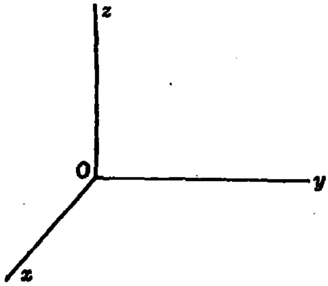

Fig. 1-1

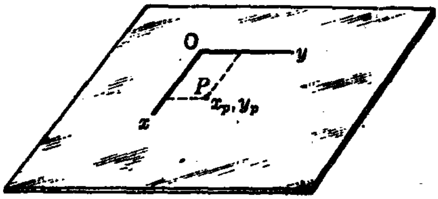

Fig. 1-2

Its coordinates are the particular values, x_p and y_p, of x and y. In this case, in which the path is restricted to one plane and the system consists of but one particle, the system is completely specified by the knowledge, as functions of time, of the two coordinates, x and y, of the particle. If two particles were present, both confined to the plane, the specification of the system would require four coordinates, the x and y coordinates of each particle. The removal of the restriction confining the path of the two particles to the plane would require the addition of the z coordinate for each particle.

When the motion is not permitted to extend freely in three dimensions, the system is said to be subject to constraints. In the instance of the particles on the table top there exists one constraint the equation of which is $z = 0$ for all time. This is a particular example of the general condition expressed by

$$f(x,y,z) = 0 \tag{1-1}$$

A constraint described by Eq. (1-1) is called an *integrable constraint.* The term *integrable* is employed here since the differential relation expressing the fact that z is not allowed to vary is $dz = 0$, an expression which is readily integrated to $z =$ const. The constant of integration is zero in the present example since the table top is in the plane $z = 0$. Relations such as Eq. (1-1) enable one, at the outset of a problem, to reduce the number of distinct coordinates which are required in order to describe the system involved in the problem. The number of coordinates eliminated is just equal to the number of the relations of the type of Eq. (1-1) which may be present.

Constraints also exist which are of the *nonintegrable* type, that is, the equations of these limitations involve differential coefficients in a manner such that they cannot be integrated. Consequently no coordinates may be eliminated by means of these relationships. Attention will be called to these again later in the text (cf. Sec. 13-1).

Simultaneously with the choice of a coordinate system careful attention must be paid to its state of motion. In Sec. 1-20 some of the complications attending an injudicious selection of coordinates will be considered.

1-3. Linear Velocity and Acceleration. Consider a particle which is experiencing a rectilinear displacement from O to P along the path shown in Fig. 1-3. At a time t the particle is at a distance s measured from point O along this path. During the subsequent increment Δt of time the particle moves through a distance Δs. The quantity $\Delta s/\Delta t$ is called the *average velocity* (time average) during the interval Δt. The *instantaneous linear velocity* at point s is then defined as

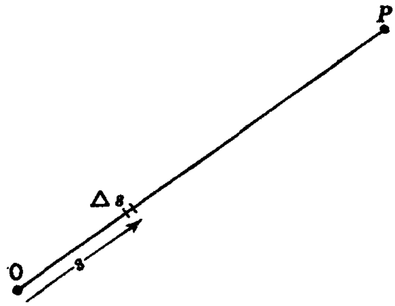

Fig. 1-3

$$v_s = \lim_{t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt} = \dot{s} \tag{1-2}$$

In Eq. (1-2) the symbol $\dot{s}$ ($= ds/dt$) has been introduced. It is read "s dot." Extensive use will be made of this notation. The term *speed* is often employed to denote the magnitude of the velocity.

A second kinematical quantity which requires definition is the *acceleration*. Suppose that at point s the particle has a velocity v_s. During the time Δt thereafter, the velocity changes by an amount Δv_s. The *average acceleration* during this interval Δt is thus $\Delta v_s/\Delta t$, from which we are immediately able to define the *instantaneous linear acceleration* at point s to be

$$a_s = \lim_{\Delta t \to 0} \frac{\Delta v_s}{\Delta t} = \frac{dv_s}{dt} = \frac{d^2 s}{dt^2} = \ddot{s}$$

Here several equivalent symbols for the acceleration have been stated. The quantity $\ddot{s}$, for example, is read "s double dot."

It is sometimes convenient to employ the terms *average velocity* and *average acceleration* in the larger sense of being the time average of these quantities during the entire time T of the translation from O to P. These averages are

$$\bar{v}_s = \frac{OP}{T} \qquad \bar{a}_s = \frac{v_P - v_O}{T} \tag{1-3}$$

where in each case the bar signifies that the time average is meant. In the second of Eqs. (1-3) v_O is the velocity in the s direction at point O (the initial velocity), and v_P is the velocity in the s direction at point P.

The terms *uniform velocity* and *uniform acceleration* will also be encountered. By *uniform* velocity and acceleration is meant that the magnitudes and directions of these quantities remain constant throughout the motion.

Example 1-1. A particle starts toward D, from rest at point A (Fig. 1-4). During the part AB of the path (a distance x_1) the particle has a uniform acceleration a_1, during the time when the particle is between B and C (a distance x_2) there is no acceleration, and during the third interval, between C and D, there is a uniform acceleration $-a_3$, where a_3 is a positive quantity. The negative sign signifies that the acceleration is in the direction of decreasing x, that is, a deceleration. The magnitude of a_3 is such that the particle will just be brought to rest at D. At what times will the particle arrive at points B, C, and D?

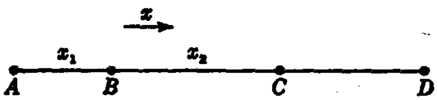

FIG. 1-4

We note first that the known quantities in the problem are a_1, a_3, x_1, and x_2, and the end results must be expressed in terms of these. For the first step of the motion the origin of x is chosen to be at A, where the particle is located at $t = 0$. The equation of motion is

$$\ddot{x} = a_1 \tag{1-4}$$

Integrating once with respect to time, we have

$$\dot{x} = a_1t + c_1 \tag{1-5}$$

where c_1 is a constant of integration. The constant c_1 can be determined from the boundary condition that at $t = 0$ the velocity $\dot{x} = 0$ also. Thus $c_1 = 0$. Integrating a second time, we obtain

$$x = \frac{a_1t^2}{2} \tag{1-6}$$

the constant of integration again being zero since $x = 0$ at $t = 0$. Thus t_1, the time required by the particle to traverse the distance x_1, becomes

$$t_1 = \sqrt{\frac{2x_1}{a_1}} \tag{1-7}$$

Similarly the time t_2 required to pass from B to C can easily be found since the velocity is unchanged during the interval $BC(= x_2)$. From Eqs. (1-5) and (1-7)

$$\dot{x}_B = a_1t_1 = \sqrt{2a_1x_1} \tag{1-8}$$

Thus the total time required by the particle to go from A to C is

$$t_{AC} = t_1 + t_2 = \sqrt{\frac{2x_1}{a_1}} + \frac{x_2}{\sqrt{2a_1x_1}} \tag{1-9}$$

In the third interval the motion is governed by the equation

$$\ddot{x} = -a_3 \tag{1-10}$$

from which, selecting new origins of x and t,

$$\dot{x} = -a_3t + \dot{x}_B = -a_3t + \sqrt{2a_1x_1}$$

and

$$t_3 = \frac{\sqrt{2a_1x_1}}{a_3}$$

and where the second equality follows since at $t = t_3$ the velocity is zero. Thus the total time t_{AD} required by the particle to pass from A to D becomes

$$t_{AD} = t_1 + t_2 + t_3 = \sqrt{\frac{2x_1}{a_1}} + \frac{x_2}{\sqrt{2a_1x_1}} + \frac{\sqrt{2a_1x_1}}{a_3} \tag{1-11}$$

1-4. Angular Velocity and Acceleration. In a manner very similar to that for the corresponding linear quantities we are able to define *angular* velocity and acceleration. Consider a particle which suffers a translation along a segment AB of a circle with center at O in Fig. 1-5. During this translation it undergoes a displacement through $\angle AOB$. We may define in the same way, as for the linear case, time rates at which the angular displacement is carried out. Thus at any angle θ the *angular velocity* (or simply *angular speed*, if the magnitude alone is being referred to) in radians per second and the *angular acceleration* in radians per second per second are defined, respectively, as

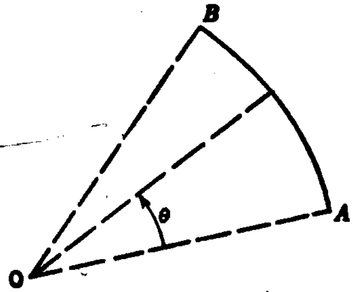

FIG. 1-5

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} = \dot{\theta} \qquad \alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \ddot{\theta} \tag{1-12}$$

ELEMENTS OF VECTOR ANALYSIS

1-5. Vectors and Scalars.[1] Two[2] classes of quantities are of great importance in elementary mechanics. These are *vectors* and *scalars*. A

[1] It was perceived very early (cf. "The Collected Works of J. Willard Gibbs", Vol. II, Longmans, Green & Co., Inc., New York, 1928) that certain physical quantities could be represented by directed segments having definite components in a given coordinate system. The relations among these directed segments themselves, rather than their components, in many cases furnished expressions of physical laws which did not depend upon any one coordinate system, a noteworthy advance indeed. A notation was developed, and the rules of manipulation of these quantities were worked out. The resulting framework is what is now known as *vector analysis*. Mathematicians have since put these procedures on a more rigorous basis.

[2] In certain more advanced physical problems the two notions of vectors and scalars