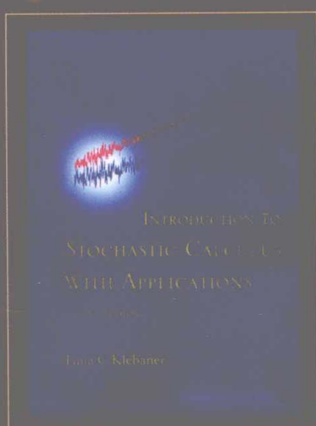
TURING 图灵原版数学·统计学系列 26

Introduction to Stochastic Calculus with Applications

随机分析及应用

（英文版·第2版）

[澳] Fima C Klebaner 著



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内 容 提 要

本书介绍了随机分析的理论和应用两方面的知识. 内容涉及积分和概率论的基础知识、基本的随机过程、布朗运动和伊藤过程的积分、随机微分方程、半鞅积分、纯离散过程，以及随机分析在金融、生物、工程和物理等方面的应用. 书中有大量的例题和习题，并附有答案，便于读者进行深层次的学习.

本书非常适合初学者阅读，可作为高等院校经管、理工和社科类各专业高年级本科生和研究生随机分析和金融数学的教材，也可供相关领域的科研人员参考.

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Preface

Preface to the Second Edition

The second edition is revised, expanded and enhanced. This is now a more complete text in Stochastic Calculus, from both a theoretical and an applications point of view. Changes came about, as a result of using this book for teaching courses in Stochastic Calculus and Financial Mathematics over a number of years. Many topics are expanded with more worked out examples and exercises. Solutions to selected exercises are included. A new chapter on bonds and interest rates contains derivations of the main pricing models, including currently used market models (BGM). The change of numeraire technique is demonstrated on interest rate, currency and exotic options. The presentation of Applications in Finance is now more comprehensive and self-contained. The models in Biology introduced in the new edition include the age-dependent branching process and a stochastic model for competition of species. These Markov processes are treated by Stochastic Calculus techniques using some new representations, such as a relation between Poisson and Birth-Death processes. The mathematical theory of filtering is based on the methods of Stochastic Calculus. In the new edition, we derive stochastic equations for a non-linear filter first and obtain the Kalman-Bucy filter as a corollary. Models arising in applications are treated rigorously demonstrating how to apply theoretical results to particular models. This approach might not make certain places easy reading, however, by using this book, the reader will accomplish a working knowledge of Stochastic Calculus.

Preface to the First Edition

This book aims at providing a concise presentation of Stochastic Calculus with some of its applications in Finance, Engineering and Science.

During the past twenty years, there has been an increasing demand for tools and methods of Stochastic Calculus in various disciplines. One of the greatest demands has come from the growing area of Mathematical Finance, where Stochastic Calculus is used for pricing and hedging of financial derivatives, such as options. In Engineering, Stochastic Calculus is used in filtering and control theory. In Physics, Stochastic Calculus is used to study the effects of random excitations on various physical phenomena. In Biology, Stochastic Calculus is used to model the effects of stochastic variability in reproduction and environment on populations.

From an applied perspective, Stochastic Calculus can be loosely described as a field of Mathematics, that is concerned with infinitesimal calculus on non-differentiable functions. The need for this calculus comes from the necessity to include unpredictable factors into modelling. This is where probability comes in and the result is a calculus for random functions or stochastic processes.

This is a mathematical text, that builds on theory of functions and probability and develops the martingale theory, which is highly technical. This text is aimed at gradually taking the reader from a fairly low technical level to a sophisticated one. This is achieved by making use of many solved examples. Every effort has been made to keep presentation as simple as possible, while mathematically rigorous. Simple proofs are presented, but more technical proofs are left out and replaced by heuristic arguments with references to other more complete texts. This allows the reader to arrive at advanced results sooner. These results are required in applications. For example, the change of measure technique is needed in options pricing; calculations of conditional expectations with respect to a new filtration is needed in filtering. It turns out that completely unrelated applied problems have their solutions rooted in the same mathematical result. For example, the problem of pricing an option and the problem of optimal filtering of a noisy signal, both rely on the martingale representation property of Brownian motion.

This text presumes less initial knowledge than most texts on the subject (Métivier (1982), Dellacherie and Meyer (1982), Protter (1992), Liptser and Shiryayev (1989), Jacod and Shiryayev (1987), Karatzas and Shreve (1988), Stroock and Varadhan (1979), Revuz and Yor (1991), Rogers and Williams (1990)), however it still presents a fairly complete and mathematically rigorous treatment of Stochastic Calculus for both continuous processes and processes with jumps.

A brief description of the contents follows (for more details see the Table of Contents). The first two chapters describe the basic results in Calculus and Probability needed for further development. These chapters have examples but no exercises. Some more technical results in these chapters may be skipped and referred to later when needed.

In Chapter 3, the two main stochastic processes used in Stochastic Calculus are given: Brownian motion (for calculus of continuous processes) and Poisson process (for calculus of processes with jumps). Integration with respect to Brownian motion and closely related processes (Itô processes) is introduced in Chapter 4. It allows one to define a stochastic differential equation. Such equations arise in applications when random noise is introduced into ordinary differential equations. Stochastic differential equations are treated in Chapter 5. Diffusion processes arise as solutions to stochastic differential equations, they are presented in Chapter 6. As the name suggests, diffusions describe a real physical phenomenon, and are met in many real life applications. Chapter 7 contains information about martingales, examples of which are provided by Itô processes and compensated Poisson processes, introduced in earlier chapters. The martingale theory provides the main tools of stochastic calculus. These include optional stopping, localization and martingale representations. These are abstract concepts, but they arise in applied problems, where their use is demonstrated. Chapter 8 gives a brief account of calculus for most general processes, called semimartingales. Basic results include Itô's formula and stochastic exponential. The reader has already met these concepts in Brownian motion calculus given in Chapter 4. Chapter 9 treats Pure Jump processes, where they are analyzed by using compensators. The change of measure is given in Chapter 10. This topic is important in options pricing, and for inference for stochastic processes. Chapters 11-14 are devoted to applications of Stochastic Calculus. Applications in Finance are given in Chapters 11 and 12, stocks and currency options (Chapter 11); bonds, interest rates and their options (Chapter 12). Applications in Biology are given in Chapter 13. They include diffusion models, Birth-Death processes, age-dependent (Bellman-Harris) branching processes, and a stochastic version of the Lotka-Volterra model for competition of species. Chapter 14 gives applications in Engineering and Physics. Equations for a non-linear filter are derived, and applied to obtain the Kalman-Bucy filter. Random perturbations to two-dimensional differential equations are given as an application in Physics. Exercises are placed at the end of each chapter.

This text can be used for a variety of courses in Stochastic Calculus and Financial Mathematics. The application to Finance is extensive enough to use it for a course in Mathematical Finance and for self study. This text is suitable for advanced undergraduate students, graduate students as well as research workers and practioners.

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Melbourne, 2004.

Contents

Chapter 1

Preliminaries From Calculus

Stochastic calculus deals with functions of time t, $0 \leq t \leq T$. In this chapter some concepts of the infinitesimal calculus used in the sequel are given.

1.1 Functions in Calculus

Continuous and Differentiable Functions

A function g is called continuous at the point $t = t_0$ if the increment of g over small intervals is small,

$$\Delta g(t) = g(t) - g(t_0) \to 0 \text{ as } \Delta t = t - t_0 \to 0.$$

If g is continuous at every point of its domain of definition, it is simply called continuous.

g is called differentiable at the point $t = t_0$ if at that point

$$\Delta g \sim C\Delta t \quad \text{or} \quad \lim_{\Delta t \to 0} \frac{\Delta g(t)}{\Delta t} = C,$$

this constant C is denoted by $g'(t_0)$. If g is differentiable at every point of its domain, it is called differentiable.

An important application of the derivative is a theorem on finite increments.

Theorem 1.1 (Mean Value Theorem) *If f is continuous on $[a, b]$ and has a derivative on (a, b), then there is c, $a < c < b$, such that*

$$f(b) - f(a) = f'(c)(b - a). \tag{1.1}$$

Clearly, differentiability implies continuity, but not the other way around, as continuity states that the increment Δg converges to zero together with Δt, whereas differentiability states that this convergence is at the same rate or faster.

Example 1.1: The function $g(t) = \sqrt{t}$ is not differentiable at 0, as at this point

$$\frac{\Delta g}{\Delta t} = \frac{\sqrt{\Delta t}}{\Delta t} = \frac{1}{\sqrt{\Delta t}} \to \infty$$

as $t \to 0$.

It is surprisingly difficult to construct an example of a continuous function which is not differentiable at *any* point.

Example 1.2: An example of a continuous, nowhere differentiable function was given by the Weierstrass in 1872: for $0 \le t \le 2\pi$

$$f(t) = \sum_{n=1}^{\infty} \frac{\cos(3^n t)}{2^n}. \tag{1.2}$$

We don't give a proof of these properties, a justification for continuity is given by the fact that if a sequence of continuous functions converges uniformly, then the limit is continuous; and a justification for non-differentiability can be provided in some sense by differentiating term by term, which results in a divergent series.

To save repetition the following notations are used: *a continuous function f is said to be a C function; a differentiable function f with continuous derivative is said to be a C^1 function; a twice differentiable function f with continuous second derivative is said to be a C^2 function; etc.*

Right and Left-Continuous Functions

We can rephrase the definition of a continuous function: a function g is called continuous at the point $t = t_0$ if

$$\lim_{t \to t_0} g(t) = g(t_0), \tag{1.3}$$

it is called right-continuous (left-continuous) at t_0 if the values of the function $g(t)$ approach $g(t_0)$ when t approaches t_0 from the right (left)

$$\lim_{t \downarrow t_0} g(t) = g(t_0), \quad (\lim_{t \uparrow t_0} g(t) = g(t_0).) \tag{1.4}$$

If g is continuous it is, clearly, both right and left-continuous.

The left-continuous version of g, denoted by $g(t-)$, is defined by taking left limit at each point,

$$g(t-) = \lim_{s \uparrow t} g(s). \tag{1.5}$$

From the definitions we have: g is left-continuous if $g(t) = g(t-)$.

The concept of $g(t+)$ is defined similarly,

$$g(t+) = \lim_{s \downarrow t} g(s). \tag{1.6}$$

If g is a right-continuous function then $g(t+) = g(t)$ for any t, so that $g_+ = g$.

Definition 1.2 *A point t is called a discontinuity of the first kind or a jump point if both limits $g(t+)$ and $g(t-)$ exist and are not equal. The jump at t is defined as $\Delta g(t) = g(t+) - g(t-)$. Any other discontinuity is said to be of the second kind.*

Example 1.3: The function $\sin(1/t)$ for $t \neq 0$ and 0 for $t = 0$ has discontinuity of the second kind at zero, because the limits from the right or the left don't exist.

An important result is that a function can have at most countably many jump discontinuities (see for example Hobson (1921), p.286).

Theorem 1.3 *A function defined on an interval $[a, b]$ can have no more than countably many jumps.*

A function, of course, can have more than countably many discontinuities, but then they are not all jumps, i.e. would not have limits from right or left.

Another useful result is that a derivative cannot have jump discontinuities at all.

Theorem 1.4 *If f is differentiable with a finite derivative $f'(t)$ in an interval, then at all points $f'(t)$ is either continuous or has a discontinuity of the second kind.*

PROOF: If t is such that $f'(t+) = \lim_{s \downarrow t} f'(s)$ exists (finite or infinite), then by the Mean Value Theorem the same value is taken by the derivative from the right

$$f'(t) = \lim_{\Delta t \downarrow 0} \frac{f(t + \Delta) - f(t)}{\Delta} = \lim_{\Delta \downarrow 0, 0 < c < \Delta} f'(c) = f'(t+).$$

Similarly for the derivative from the left, $f'(t) = f'(t-)$. Hence $f'(t)$ is continuous at t. The result follows.

□

This result explains why functions with continuous derivatives are sought as solutions to ordinary differential equations.

Functions considered in Stochastic Calculus

Functions considered in stochastic calculus are functions without discontinuities of the second kind, that is functions that have both right and left limits at any point of the domain and have one-sided limits at the boundary. These functions are called *regular* functions. It is often agreed to identify functions if they have the same right and left limits at any point.

The class $D = D[0,T]$ of right-continuous functions on $[0,T]$ with left limits has a special name, *càdlàg* functions (which is the abbreviation of "right continuous with left limits" in French). Sometimes these processes are called R.R.C. for regular right continuous. Notice that this class of processes includes C, the class of continuous functions.

Let $g \in D$ be a càdlàg function, then by definition, all the discontinuities of g are jumps. By Theorem 1.3 such functions have no more than countably many discontinuities.

Remark 1.1: In stochastic calculus $\Delta g(t)$ usually stands for the size of the jump at t. In standard calculus $\Delta g(t)$ usually stands for the increment of g over $[t, t+\Delta]$, $\Delta g(t) = g(t+\Delta) - g(t)$. The meaning of $\Delta g(t)$ will be clear from the context.

1.2 Variation of a Function

If g is a function of real variable, its variation over the interval $[a,b]$ is defined as

$$V_g([a,b]) = \sup \sum_{i=1}^{n} |g(t_i^n) - g(t_{i-1}^n)|, \tag{1.7}$$

where the supremum is taken over partitions:

$$a = t_0^n < t_1^n < \ldots < t_n^n = b. \tag{1.8}$$

Clearly, (by the triangle inequality) the sums in (1.7) increase as new points are added to the partitions. Therefore variation of g is

$$V_g([a,b]) = \lim_{\delta_n \to 0} \sum_{i=1}^{n} |g(t_i^n) - g(t_{i-1}^n)|, \tag{1.9}$$

where $\delta_n = \max_{1 \le i \le n}(t_i - t_{i-1})$. If $V_g([a,b])$ is finite then g is said to be a function of finite variation on $[a,b]$. If g is a function of $t \geq 0$, then the variation function of g as a function of t is defined by

$$V_g(t) = V_g([0,t]).$$

Clearly, $V_g(t)$ is a non-decreasing function of t.

Definition 1.5 *g is of finite variation if $V_g(t) < \infty$ for all t. g is of bounded variation if $\sup_t V_g(t) < \infty$, in other words, if for all t, $V_g(t) < C$, a constant independent of t.*

Example 1.4:

1. If $g(t)$ is increasing then for any i, $g(t_i) > g(t_{i-1})$ resulting in a telescoping sum, where all the terms excluding the first and the last cancel out, leaving
$$V_g(t) = g(t) - g(0).$$
2. If $g(t)$ is decreasing then, similarly,
$$V_g(t) = g(0) - g(t).$$

Example 1.5: If $g(t)$ is differentiable with continuous derivative $g'(t)$, $g(t) = \int_0^t g'(s)ds$, and $\int_0^t |g'(s)|ds < \infty$, then

$$V_g(t) = \int_0^t |g'(s)|ds.$$

This can be seen by using the definition and the mean value theorem. $\int_{t_{i-1}}^{t_i} g'(s)ds = g'(\xi_i)(t_i - t_{i-1})$, for some $\xi_i \in (t_{i-1}, t_i)$. Thus $|\int_{t_{i-1}}^{t_i} g'(s)ds| = |g'(\xi_i)|(t_i - t_{i-1})$, and

$$\begin{aligned} V_g(t) &= \lim \sum_{i=1}^n |g(t_i) - g(t_{i-1})| = \lim \sum_{i=1}^n |\int_{t_{i-1}}^{t_i} g'(s)ds| \\ &= \sup \sum_{i=1}^n |g'(\xi_i)|(t_i - t_{i-1}) = \int_0^t |g'(s)|ds. \end{aligned}$$

The last equality is due to the last sum being a Riemann sum for the final integral.

Alternatively, the result can be seen from the decomposition of the derivative into the positive and negative parts,

$$g(t) = \int_0^t g'(s)ds = \int_0^t [g'(s)]^+ ds - \int_0^t [g'(s)]^- ds.$$

Notice that $[g'(s)]^-$ is zero when $[g'(s)]^+$ is positive, and the other way around. Using this one can see that the total variation of g is given by the sum of the variation of the above integrals. But these integrals are monotone functions with the value zero at zero. Hence

$$\begin{aligned} V_g(t) &= \int_0^t [g'(s)]^+ ds + \int_0^t [g'(s)]^- ds \\ &= \int_0^t ([g'(s)]^+ + [g'(s)]^-)ds = \int_0^t |g'(s)|ds. \end{aligned}$$

Example 1.6: (Variation of a pure jump function).
If g is a regular right-continuous (càdlàg) function or regular left-continuous (càglàd), and changes only by jumps,

$$g(t) = \sum_{0 \le s \le t} \Delta g(s),$$

then it is easy to see from the definition that

$$V_g(t) = \sum_{0 \le s \le t} |\Delta g(s)|.$$

Example 1.7: The function $g(t) = t \sin(1/t)$ for $t > 0$, and $g(0) = 0$ is continuous on $[0, 1]$, differentiable at all points except zero, but has infinite variation on any interval that includes zero. Take the partition $1/(2\pi k + \pi/2), 1/(2\pi k - \pi/2)$, $k = 1, 2, \ldots$.

The following theorem gives necessary and sufficient conditions for a function to have finite variation.

Theorem 1.6 (Jordan Decomposition) *Any function $g : [0, \infty) \to \mathbb{R}$ of finite variation can be expressed as the difference of two increasing functions*

$$g(t) = a(t) - b(t).$$

One such decomposition is given by

$$a(t) = V_g(t) \quad b(t) = V_g(t) - g(t). \tag{1.10}$$

It is easy to check that $b(t)$ is increasing, and $a(t)$ is obviously increasing. The representation of a function of finite variation as difference of two increasing functions is not unique. Another decomposition is

$$g(t) = \frac{1}{2}(V_g(t) + g(t)) - \frac{1}{2}(V_g(t) - g(t)).$$

The sum, the difference and the product of functions of finite variation are also functions of finite variation. This is also true for the ratio of two functions of finite variation provided the modulus of the denominator is larger than a positive constant.

The following result follows by Theorem 1.3, and its proof is easy.

Theorem 1.7 *A finite variation function can have no more than countably many discontinuities. Moreover, all discontinuities are jumps.*