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Multi-Grid Methods and Applications

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With 43 Figures and 48 Tables

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Preface

Although multi-grid methods had already been described in the early 1960's, it was not until the mid-seventies that they were realised to be very efficient methods of solution with a broad area of application. Over the past ten years, the number of publications on this topic has grown rapidly with the result that in searching for information about multi-grid methods at the present time, one is confronted with an abundance of articles scattered in a wide range of periodicals and proceedings.

This monograph is an attempt to describe the basic concepts of multi-grid methods. Different groups of readers may be interested in different parts of the book. The first part, which concentrates on the algorithmic details, is intended for readers interested in the theory and practice of multi-grid methods. The second part is devoted to the mathematical analysis, and is especially intended for mathematicians. Readers interested in engineering and technical applications will find numerous chapters on specific multi-grid applications and additional techniques: the primary emphasis is on applications in the field of fluid dynamics. A special chapter is dedicated to the multi-grid algorithms for integral equations, a topic which up until now has found little recognition in the literature.

I would like to thank numerous colleagues here and abroad for the many fruitful discussions and ideas they have shared with me. My special thanks to my colleagues S. McKee, G. Shaw, and S. Trickett at Oxford for anglicising my manuscript. I would also like to thank Mrs. B. Koberling for carefully typing the major portion of the manuscript. I am very grateful to my colleagues D. Braess and R. Verfürth and my associates G. Hofmann and G. Wittum for their help in proofreading. Last but not least, I would like to thank Springer-Verlag for their friendly cooperation.

Kiel, June 1985 W. Hackbusch

Contents

1. Preliminaries

1.1 Introduction

The numerical solution of boundary value problems is indispensable in almost all fields of physics and engineering sciences. The recent development, e.g. the study of three-dimensional problems, leads to systems of a larger and larger number of equations. Although the computers have become faster and vector computers are available, new numerical methods are required. A step in this direction was the development of fast Poisson solvers in the late sixties. At that time it seemed that there exist faster numerical methods the simpler the discrete elliptic problem. The first multi-grid methods have also been applied to Poisson's equation and show an efficiency similar to that of the direct solvers. But differently from other numerical methods, the efficiency is not lost when more involved problems are to be solved.

The characteristic feature of the multi-grid iteration is its fast convergence. The convergence speed does not deteriorate when the discretisation is refined, whereas classical iterative methods slow down for decreasing grid size. As a consequence one obtains an acceptable approximation of the discrete problem at the expense of computational work proportional to the number of unknowns, which is also the number of the equations in the system. It is not only the complexity which is optimal, also the constant of proportionality is so small that other methods can hardly surpass the multi-grid efficiency.

The previous characterisation does not mean that there is a fixed multi-grid algorithm applying to all boundary value problems. There is rather a multi-grid technique fixing only the framework of the algorithm. The efficiency of the multi-grid algorithm depends on the adjustment of its components to the problem in question. Therefore it is one task of this book to describe the basic framework and the scope of several of the different multi-grid components.

The book is divided into five parts:

Part I	(§§ 2–5):	linear multi-grid algorithms
Part II	(§§ 6–8):	convergence analysis
Part III	(§§ 9–12):	special multi-grid applications
Part IV	(§§ 13–15):	additional techniques
Part V	(§§ 16):	application to integral equations

Part I presents the basic algorithmic multi-grid concept almost without theoretical considerations. The exception is § 2, where we analyse in some detail a one-dimensional problem in order to understand the characteristic behaviour of multi-grid iterations.

To contradict the impression the multi-grid algorithms are based only on heuristical considerations, Part II is devoted to a comprehensive convergence analysis.

The adaptation of multi-grid algorithms to more general elliptic problems is described in Part III. Here the reader finds applications to nonlinear problems, to convection diffusion equations, and to the important Stokes and Navier-Stokes equations.

In Part IV we suggest numerical techniques which can easily be combined with the multi-grid solution process to yield an extra efficiency.

Less attention has been paid to the application of multi-grid algorithms to integral equations. As described in Part V the extremely fast multi-grid iteration of the second kind applies not only to common integral equations but also to more general equations of the second kind, which leads to several interesting applications.

1.2 Notation

The formulae are numbered separately in each subsection. For instance, the tenth formula of Subsection 3.1 is denoted by (3.1.10). Within the same subsection we refer to this formula as

(10),

whereas we write

(1.10)

in other parts of Sect. 3. The full notation

(3.1.10)

is used outside of Sect. 3.

Below we list those symbols which have a special meaning throughout the book.

a, $a(\cdot,\cdot)$	sesqui-linear form; see § 1.4.5
$\mathbb{C}$	set of complex numbers
C, C_0, C_1, etc.	generic constants
$C^s(\Omega)$	Hölder spaces; see § 1.4.1
d	dimension of Ω; see § 3.1.1
f, f_l	right-hand sides; see § 1.3.1, § 3.1.1

$\mathscr{F}_l$	linear space of right-hand functions; see §6.1
$\mathscr{F}_l(\varrho)$	see §9.5.1
h_l	discretisation parameter (grid size) at level l; see §2.1
$\mathscr{H}, \mathscr{H}^1$	Hilbert space, on which the variational formulation is based; see §3.1.3, §6.3.1.1
$\mathscr{H}^s$	see §6.3.1.1
$\mathscr{H}_l$	finite element subspace of $\mathscr{H}$; see §3.1.3
$H^k(\Omega)$, $H_0^k(\Omega)$, $H^s(\Omega)$, etc.	Sobolev spaces; see §1.4.2
I	identity mapping, identity matrix
l	level number; see §2.1
L, L_Ω, L_Γ	operators associated with the boundary value problem; see §3.1.1, §3.1.3
L_l	matrix of the linear system; see §1.3.1
L_l, $L_l(v_l)$	Jacobian of $\mathscr{L}_l$ (at v_l); see §9.1
$\mathscr{L}_l$, $\mathscr{L}_l(u_l)$	nonlinear operator at level l; see §9.1
m	$2m$ is the order of the differential operator; see §3.1.1
M_l, M_l', $M_l(v_1, v_2)$, etc.	iteration matrix (in particular of the two- and multi-grid iteration); see §1.3.1, §6.1.1
$\mathbb{N}$	set of natural numbers
N_l	see §1.3.1
$\mathscr{N}_l$, $\mathscr{N}_l(\lambda)$, $\mathscr{N}_l^*$, etc.	eigenspaces; see §12.1.1
n_l	number of variables at level l; see §4.3
0	zero matrix
$O(\cdot)$, $o(\cdot)$	Landau symbols
p	prolongation from level $l-1$ to l; see §2.3
$\tilde{p}$	see §5.1
P_l	prolongation P_l: $\mathscr{U}_l \to \mathscr{H}_l$ in the finite element case; see §3.1.3, §3.6
$\hat{P}_l$	see §3.6
r	restriction r: $\mathscr{F}_l \to \mathscr{F}_{l-1}$; see §2.3
$\tilde{r}$	restriction $\tilde{r}$: $\mathscr{U}_l \to \mathscr{U}_{l-1}$; see §9.3.2
r_{inj}	trivial injection; see §3.5
$\mathbb{R}$	set of real numbers
R_l	restriction R_l: $\mathscr{H}' \to \mathscr{F}_l$; see §3.6
$\hat{R}_l$	see §3.6
$\tilde{R}_l$	restriction $\tilde{R}_l$: $\mathscr{U} \to \mathscr{U}_l$; see §14.1
$\mathscr{S}_l$, $\mathscr{S}_l^{(\nu)}$, $\mathscr{S}_l^\nu$	linear or nonlinear smoothing method; see §2.2, §3.2

S_l	iteration matrix of $\mathcal{S}_l$; see § 2.2
$s_{\mathcal{F}}, s_{\mathcal{U}}$	see (6.2.7)
T_l	see § 2.2
$\mathcal{T}_l$	finite element triangulation; see § 3.8
u_l	discrete solution; see § 1.3.1, § 9.1
$\tilde{u}_l$	approximation to u_l
u_l^i	i^{th} iterate; see § 1.3.1
$\mathcal{U}_l(\varrho)$	see § 9.5.1
$\mathbf{x}, \mathbf{x}', \mathbf{y}$ etc.	spatial vectors in $\mathbb{R}^d$
(x, y)	coordinates in $\mathbb{R}^2$
$\mathbb{Z}$	set of integers
α	exponent involved in the smoothing and approximation properties; see § 6.1.3
γ	number of iterations on the coarser grid; see § 4.1
Γ	boundary of Ω
ζ, ζ_ν	contraction number (i.e. bound on the norm of the iteration matrix); see § 1.3.2, § 2.4
$\eta(\nu)$	function involved in the smoothing property; see § 6.1.3
$\eta_0(s)$	function defined in (6.2.1)
κ	often: consistency order; see § 5.2
ν	number of smoothing iterations; see § 3.2
ν_1	number of pre-smoothing iterations
ν_2	number of post-smoothing iterations
$\varrho(A)$	spectral radius of A; see § 1.3.1
ϱ, ϱ_ν	bound on the convergence rate; see § 2.4
ϱ_B	smoothing rate; see § 2.6.3, § 8.2.3
ϱ_L	smoothing number; see § 2.6.3
$\sigma(A)$	spectrum of A; see § 1.3.1
ω, ω_l	relaxation factor; see § 3.3
$\boldsymbol{\omega}$	vector of relaxation factors for semi-iterative methods; see § 3.3.5
Ω	domain of the boundary value problem
Ω_l	grid at level l
$\Omega_l^i, \Omega_l^r, \Omega_l^b$	subsets of Ω_l; see § 3.3.1
$(\cdot,\cdot)_{\mathcal{U}}, \langle\cdot,\cdot\rangle_{\mathcal{U}}$	scalar product in a Hilbert space $\mathcal{U}$; see § 1.4.5
$(\cdot,\cdot)$	scalar product, in particular, of $L^2(\Omega)$
$\langle\cdot,\cdot\rangle$	scalar product in $\mathcal{U}_l$; see § 3.5

$\|\cdot\|_{\mathscr{U}}$, $\|\cdot\|_{\mathscr{F}}$	norms of $\mathscr{U}_l$ and $\mathscr{F}_l$, respectively; see §6.1.1
$\lvert\cdot\rvert_0$	Euclidean norm on $\mathscr{U}_l$ and $\mathscr{F}_l$; see §1.3.2, §6.2.1
$\lvert\cdot\rvert_s$	discrete Sobolev norm of order sm; see §2.6.2, §6.2.1
$\|\cdot\|_{\mathscr{F}\leftarrow\mathscr{U}}$, $\|\cdot\|_{\mathscr{U}\leftarrow\mathscr{U}}$, etc.	matrix norms associated with $\|\cdot\|_{\mathscr{U}}$ and $\|\cdot\|_{\mathscr{F}}$; see §1.3.2
$\lvert\cdot\rvert_{s\leftarrow t}$	matrix norm associated with $\lvert\cdot\rvert_s$ and $\lvert\cdot\rvert_t$; see §6.2.1
$\|\cdot\|$	spectral norm ($\|\cdot\| = \lvert\cdot\rvert_{0\leftarrow 0}$); see §1.3.2

1.3 Some Elements from Linear Algebra

For the sake of completeness some notation, definitions, and statements from the field of linear algebra are given. In the first subsection we recall some results concerning linear iterations. Thereafter we remind the reader of the definition of norms, matrix norms, and scalar products.

1.3.1 Analysis of Iterative Processes

Let $\mathscr{U}_l$ be an n_l-dimensional vector space[1] and consider a system

$$L_l u_l = f_l \qquad (u_l, f_l \in \mathscr{U}_l) \tag{1.3.1}$$

of n_l linear equations. An iteration for solving Eq. (1) is a process $u_l^0 \mapsto u_l^1 \mapsto \ldots \mapsto u_l^{j-1} \mapsto u_l^j \mapsto \ldots$ generated by some linear mapping φ_l:

$$u_l^{j+1} = \varphi_l(u_l^j, f_l).$$

The multi-grid iteration and also the well-known classical iterations are of this form. As φ_l is linear it can be represented by $\varphi_l(u_l, f_l) = M_l u_l + N_l f_l$. The iteration becomes

$$u_l^{j+1} = M_l u_l^j + N_l f_l. \tag{1.3.2}$$

M_l is called the iteration matrix. The explicit representation of the j^{th} iterate u_l^j is

$$u_l^j = M_l^j u_l^0 + N_l^{(j)} f_l, \qquad N_l^{(j)} = \sum_{k=0}^{j-1} M_l^k N_l. \tag{1.3.3}$$

An obvious condition on the iteration (2) is that the solution of (1) is a fixed point of (2):

$$u_l = M_l u_l + N_l f_l \qquad (u_l \text{ is the solution of (1)}). \tag{1.3.4}$$

1 Usually, we shall consider real vector spaces. Complex spaces are needed only in connection with Fourier transforms