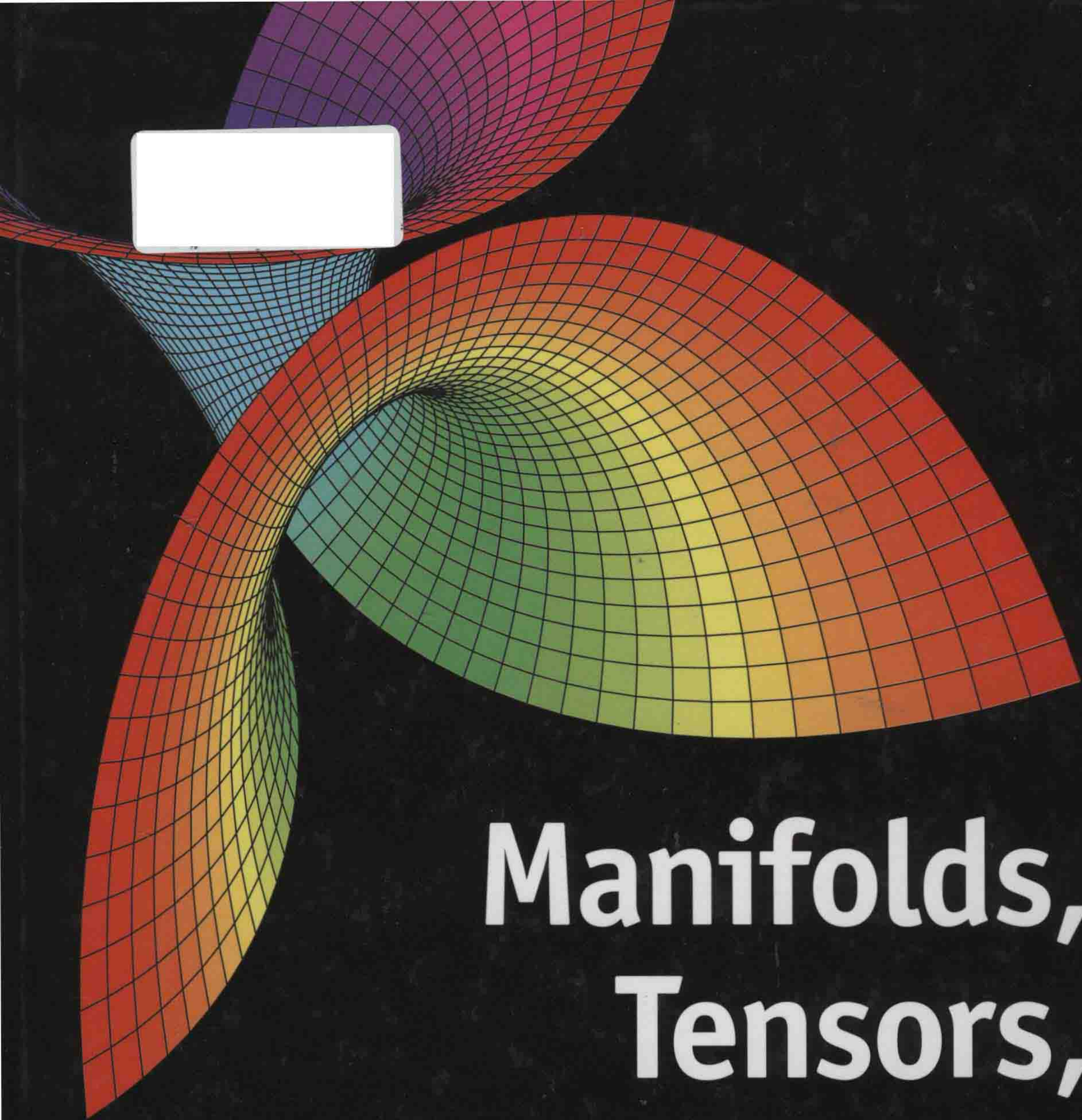

Manifolds, Tensors, and Forms

An Introduction for Mathematicians and Physicists

PAUL RENTELN

CAMBRIDGE

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and
California Institute of Technology

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Providing a succinct yet comprehensive treatment of the essentials of modern differential geometry and topology, this book's clear prose and informal style make it accessible to advanced undergraduate and graduate students in mathematics and the physical sciences.

The text covers the basics of multilinear algebra, differentiation and integration on manifolds, Lie groups and Lie algebras, homotopy and de Rham cohomology, homology, vector bundles, Riemannian and pseudo-Riemannian geometry, and degree theory. It also features over 250 detailed exercises, and a variety of applications revealing fundamental connections to classical mechanics, electromagnetism (including circuit theory), general relativity, and gauge theory. Solutions to the problems are available for instructors at www.cambridge.org/9781107042193.

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Preface

> Q: What's the difference between an argument and a proof? A: An argument will convince a reasonable person, but a proof is needed to convince an unreasonable one.
>
> *Anon.*

> Die Mathematiker sind eine Art Franzosen: Redet man zu ihnen, so bersetzen sie es in ihre Sprache, und dann ist es alsbald ganz etwas anderes. (Mathematicians are like Frenchmen: whatever you say to them they translate into their own language and forthwith it is something entirely different.)
>
> *Johann Wolfgang von Goethe*

This book offers a concise overview of some of the main topics in differential geometry and topology and is suitable for upper-level undergraduates and beginning graduate students in mathematics and the sciences. It evolved from a set of lecture notes on these topics given to senior-year students in physics based on the marvelous little book by Flanders [25], whose stylistic and substantive imprint can be recognized throughout. The other primary sources used are listed in the references.

By intent the book is akin to a whirlwind tour of many mathematical countries, passing many treasures along the way and only stopping to admire a few in detail. Like any good tour, it supplies all the essentials needed for individual exploration after the tour is over. But, unlike many tours, it also provides language instruction. Not surprisingly, most books on differential geometry are written by mathematicians. This one is written by a mathematically inclined physicist, one who has lived and worked on both sides of the linguistic and formalistic divide that often separates pure and applied mathematics. It is this language barrier that often causes the beginner so much trouble when approaching the subject for the first time. Consequently, the book has been written with a conscious attempt to explain as much as possible from both a "high brow" and a "low brow" viewpoint,[1] particularly in the early chapters.

For many mathematicians, mathematics is the art of avoiding computation. Similarly, physicists will often say that you should never begin a computation unless you know what the answer will be. This may be so, but, more often than not, what happens is that a person works out the answer by ugly computation, and then reworks and publishes the answer in a way that hides all the gory details and makes it seem as though he or she knew the answer all along from pure abstract thought. Still, it is true that there are times when an answer can be obtained much more easily by means of a powerful abstract tool. For this reason, both approaches are given their due here. The result is a compromise between highly theoretical approaches and concrete calculational tools.

This compromise is evident in the use of proofs throughout the book. The one thing that unites mathematicians and scientists is the desire to know, not just *what* is true, but *why* it is true. For this reason, the book contains both proofs and computations. But, in the spirit of the above quotation, arguments sometimes substitute for formal proofs and many long, tedious proofs have been omitted to promote the flow of the exposition. The book therefore risks being not mathematically rigorous enough for some readers and too much so for others, but its virtue is that it is neither encylopedic nor overly pedantic. It is my hope that the presentation will appeal to readers of all backgrounds and interests.

The pace of this work is quick, hitting only the highlights. Although the writing is deliberately terse the tone of the book is for the most part informal, so as to facilitate its use for self-study. Exercises are liberally sprinkled throughout the text and sometimes referred to in later sections; additional problems are placed at the end of most chapters.[2] Although it is not necessary to do all of them, it is certainly advisable to do some; in any case you should read them all, as they provide flesh for the bare bones. After working through this book a student should have acquired all the tools needed to use these concepts in scientific applications. Of course, many topics are omitted and every major topic treated here has many books devoted to it alone. Students wishing to fill in the gaps with more detailed investigations are encouraged to seek out some of the many fine works in the reference list at the end of the book.

[1] The playful epithets are an allusion, of course, to modern humans (abstract thinkers) and Neanderthals (concrete thinkers).

[2] A solutions manual is available to instructors at www.cambridge.org/9781107042193.

The prerequisites for this book are solid first courses in linear algebra, multivariable calculus, and differential equations. Some exposure to point set topology and modern algebra would be nice, but it is not necessary. To help bring students up to speed and to avoid the necessity of looking elsewhere for certain definitions, a mathematics primer is included as Appendix A. Also, the beginning chapter contains all the linear algebra facts employed elsewhere in the book, including a discussion of the correct placement and use of indices.[3] This is followed by a chapter on tensors and multilinear algebra in preparation for the study of tensor analysis and differential forms on smooth manifolds. The de Rham cohomology leads naturally into the topology of smooth manifolds, and from there to a rather brief chapter on the homology of continuous manifolds. The tools introduced there provide a nice way to understand integration on manifolds and, in particular, Stokes' theorem, which is afforded two kinds of treatment. Next we consider vector bundles, connections, and covariant derivatives and then manifolds with metrics. The last chapter offers a very brief introduction to degree theory and some of its uses. This is followed by several appendices providing background material, calculations too long for the main body of the work, or else further applications of the theory.

Originally the book was intended to serve as the basis for a rapid, one-quarter introduction to these topics. But inevitably, as with many such projects, it began to suffer from mission creep, so that covering all the material in ten weeks would probably be a bad idea. Instructors laboring under a short deadline can, of course, simply choose to omit some topics. For example, to get to integration more quickly one could skip Chapter 5 altogether, then discuss only version two of Stokes' theorem. Instructors having the luxury of a semester system should be able to cover everything. Starred sections can be (or perhaps ought to be) skimmed or omitted on a first reading.

This is an expository work drawing freely from many different sources (most of which are listed in the references section), so none of the material harbors any pretense of originality. It is heavily influenced by lectures of Bott and Chern, whose classes I was fortunate enough to take. It also owes a debt to the expository work of many other writers, whose contributions are hereby acknowledged. My sincere apologies to anyone I may have inadvertently missed in my attributions. The manuscript itself was originally typeset using Knuth's astonishingly versatile TeXprogram (and its offspring, LaTeX), and the figures were made using Timothy Van Zandt's wonderful graphics tool `pstricks`, enhanced by the three-dimensional drawing packages `pst-3dplot` and `pst-solides3d`.

[3] Stephen Hawking jokes in the introduction to his *Brief History of Time* that the publisher warned him his sales would be halved if he included even one equation. Analogously, sales of this book may be halved by the presence of indices, as many pure mathematicians will do anything to avoid them.

It goes without saying that all writers owe a debt to their teachers. In my case I am fortunate to have learned much from Abhay Ashtekar, Raoul Bott, Shing Shen Chern, Stanley Deser, Doug Eardley, Chris Isham, Karel Kuchař, Robert Lazarsfeld, Rainer Kurt Sachs, Ted Shifrin, Lee Smolin, and Philip Yasskin, who naturally bear all responsibility for any errors contained herein . . . I also owe a special debt to Laurens Gunnarsen for having encouraged me to take Chern's class when we were students together at Berkeley and for other very helpful advice. I am grateful to Rick Wilson and the Mathematics Department at the California Institute of Technology for their kind hospitality over the course of many years and for providing such a stimulating research environment in which to nourish my other life in combinatorics. Special thanks go to Nicholas Gibbons, Lindsay Barnes, and Jessica Murphy at Cambridge University Press, for their support and guidance throughout the course of this project, and to Susan Parkinson, whose remarkable editing skills resulted in many substantial improvements to the book. Most importantly, this work would not exist without the love and affection of my wife, Alison, and our sons David and Michael.

Contents

1

Linear algebra

Eighty percent of mathematics is linear algebra.

Raoul Bott

This chapter offers a rapid review of some of the essential concepts of linear algebra that are used in the rest of the book. Even if you had a good course in linear algebra, you are encouraged to skim the chapter to make sure all the concepts and notations are familiar, then revisit it as needed.

1.1 Vector spaces

The standard example of a vector space is $\mathbb{R}^n$, which is the Cartesian product of $\mathbb{R}$ with itself n times: $\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$. A vector v in $\mathbb{R}^n$ is an n-tuple $(a_1, a_2, \ldots, a_n)$ of real numbers with scalar multiplication and vector addition defined as follows[†]:

$$c(a_1, a_2, \ldots, a_n) := (ca_1, ca_2, \ldots, ca_n) \tag{1.1}$$

and

$$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) := (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n). \tag{1.2}$$

The zero vector is $(0, 0, \ldots, 0)$.

More generally, a **vector space** V over a field $\mathbb{F}$ is a set $\{u, v, w, \ldots\}$ of objects called **vectors**, together with a set $\{a, b, c, \ldots\}$ of elements in $\mathbb{F}$ called **scalars**, that is closed under the taking of linear combinations:

$$u, v \in V \text{ and } a, b \in \mathbb{F} \Rightarrow au + bv \in V, \tag{1.3}$$

and where $0v = 0$ and $1v = v$. (For the full definition, see Appendix A.)

[†] The notation A := B means that A is defined by B.

A **subspace** of V is a subset of V that is also a vector space. An **affine subspace** of V is a translate of a subspace of V.[1] The vector space V is the **direct sum** of two subspaces U and W, written $V = U \oplus W$, if $U \cap W = 0$ (the only vector in common is the zero vector) and every vector $v \in V$ can be written uniquely as $v = u + w$ for some $u \in U$ and $w \in W$.

A set $\{v_i\}$ of vectors[2] is **linearly independent** (over the field $\mathbb{F}$) if, for any collection of scalars $\{c_i\} \subset \mathbb{F}$,

$$\sum_i c_i v_i = 0 \qquad \text{implies} \qquad c_i = 0 \text{ for all } i. \tag{1.4}$$

Essentially this means that no member of a set of linearly independent vectors may be expressed as a linear combination of the others.

EXERCISE **1.1** Prove that the vectors $(1, 1)$ and $(2, 1)$ in $\mathbb{R}^2$ are linearly independent over $\mathbb{R}$ whereas the vectors $(1, 1)$ and $(2, 2)$ in $\mathbb{R}^2$ are linearly dependent over $\mathbb{R}$.

A set B of vectors is a **spanning set** for V (or, more simply, **spans** V) if every vector in V can be written as a linear combination of vectors from B. A spanning set of linearly independent vectors is called a **basis** for the vector space. The cardinality of a basis for V is called the **dimension** of the space, written $\dim V$. Vector spaces have many different bases, and they all have the same cardinality. (For the most part we consider only finite dimensional vector spaces.)

Example 1.1 The vector space $\mathbb{R}^n$ is n-dimensional over $\mathbb{R}$. The **standard basis** is the set of n vectors $\{e_1 = (1, 0, \dots, 0), e_2 = (0, 1, \dots, 0), \dots, e_n = (0, 0, \dots, 1)\}$.

Pick a basis $\{e_i\}$ for the vector space V. By definition we may write

$$v = \sum_i v_i e_i \tag{1.5}$$

for any vector $v \in V$, where the v_i are elements of the field $\mathbb{F}$ and are called the **components** of v with respect to the basis $\{e_i\}$. We get a different set of components for the same vector v depending upon the basis we choose.

EXERCISE **1.2** Show that the components of a vector are unique, that is, if $v = \sum_i v_i e_i = \sum_i v'_i e_i$ then $v_i = v'_i$.

EXERCISE **1.3** Show that $\dim(U \oplus W) = \dim U + \dim W$.

[1] By definition, W is a **translate** of U if, for some fixed $v \in V$ with $v \neq 0$, $W = \{u + v : u \in U\}$. An affine subspace is like a subspace without the zero vector.

[2] To avoid cluttering the formulae, the index range will often be left unspecified. In some cases this is because the range is arbitrary, while in other cases it is because the range is obvious.

EXERCISE 1.4 Let W be a subspace of V. Show that we can always **complete a basis** of W to obtain one for V. In other words, if $\dim W = m$ and $\dim V = n$, and if $\{f_1, \ldots, f_m\}$ is a basis for W, show there exists a basis for V of the form $\{f_1, \ldots, f_m, g_1, \ldots, g_{n-m}\}$. Equivalently, show that a basis of a finite-dimensional vector space is just a maximal set of linearly independent vectors.

1.2 Linear maps

Let V and W be vector spaces. A map $T : V \to W$ is **linear** (or a **homomorphism**) if, for $v_1, v_2 \in V$ and $a_1, a_2 \in \mathbb{F}$,

$$T(a_1 v_1 + a_2 v_2) = a_1 T v_1 + a_2 T v_2. \tag{1.6}$$

We will write either $T(v)$ or Tv for the action of the linear map T on a vector v.

EXERCISE 1.5 Show that two linear maps that agree on a basis agree everywhere.

Given a finite subset $U := \{u_1, u_2, \ldots\}$ of vectors in V, any map $T : U \to W$ induces a linear map $T : V \to W$ according to the rule $T(\sum_i a_i u_i) := \sum_i a_i T u_i$. The original map is said to have been **extended by linearity**.

The set of all $v \in V$ such that $Tv = 0$ is called the **kernel** (or **null space**) of T, written $\ker T$; $\dim \ker T$ is sometimes called the **nullity** of T. The set of all $w \in W$ for which there exists a $v \in V$ with $Tv = w$ is called the **image** (or **range**) of T, written $\operatorname{im} T$. The **rank** of T, $\operatorname{rk} T$, is defined as $\dim \operatorname{im} T$.

EXERCISE 1.6 Show that $\ker T$ is a subspace of V and $\operatorname{im} T$ is a subspace of W.

EXERCISE 1.7 Show that T is injective if and only if the kernel of T consists of the zero vector alone.

If T is bijective it is called an **isomorphism**, in which case V and W are said to be **isomorphic**; this is written as $V \cong W$ or, sloppily, $V = W$. Isomorphic vector spaces are not necessarily identical, but they behave as if they were.

Theorem 1.1 *All finite-dimensional vector spaces of the same dimension are isomorphic.*

EXERCISE 1.8 Prove Theorem 1.1.

A linear map from a vector space to itself is called an **endomorphism**, and if it is a bijection it is called an **automorphism**.[3]

[3] Physicists tend to call an endomorphism a **linear operator**.

EXERCISE 1.9 A linear map T is **idempotent** if $T^2 = T$. An idempotent endomorphism $\pi : V \to V$ is called a **projection (operator)**. *Remark:* This is not to be confused with an orthogonal projection, which requires an inner product for its definition.

(a) Show that $V = \operatorname{im}\pi \oplus \ker\pi$.
(b) Suppose W is a subspace of V. Show that there exists a projection operator $\pi : V \to V$ that restricts to the identity map on W. (Note that the projection operator is not unique.) *Hint:* Complete a basis of W so that it becomes a basis of V.

EXERCISE 1.10 Show that if $T : V \to V$ is an automorphism then the inverse map T^{-1} is also linear.

EXERCISE 1.11 Show that the set Aut V of all automorphisms of V is a group. (For information about groups, consult Appendix A.)

1.3 Exact sequences

Suppose that you are given a sequence of vector spaces V_i and linear maps $\varphi_i : V_i \to V_{i+1}$ connecting them, as illustrated below:

$$\cdots \longrightarrow V_{i-1} \xrightarrow{\varphi_{i-1}} V_i \xrightarrow{\varphi_i} V_{i+1} \xrightarrow{\varphi_{i+1}} \cdots .$$

The maps are said to be **exact at** V_i if $\operatorname{im}\varphi_{i-1} = \ker\varphi_i$, i.e., the image of φ_{i-1} equals the kernel of φ_i. The sequence is called an **exact sequence** if the maps are exact at V_i for all i. Exact sequences of vector spaces show up everywhere and satisfy some particularly nice properties, so it worth exploring them a bit.

If V_1, V_2, and V_3 are three vector spaces, and if the sequence

$$0 \xrightarrow{\varphi_0} V_1 \xrightarrow{\varphi_1} V_2 \xrightarrow{\varphi_2} V_3 \xrightarrow{\varphi_3} 0 \tag{1.7}$$

is exact, it is called a **short exact sequence**. In this diagram "0" represents the zero-dimensional vector space, whose only element is the zero vector. The linear map φ_0 sends 0 to the zero vector of V_1, while φ_3 sends everything in V_3 to the zero vector.

EXERCISE 1.12 Show that the existence of the short exact sequence (1.7) is equivalent to the statement "φ_1 is injective and φ_2 is surjective." In particular, if

$$0 \longrightarrow V \xrightarrow{\varphi} W \longrightarrow 0 \tag{1.8}$$

is exact, V and W must be isomorphic.