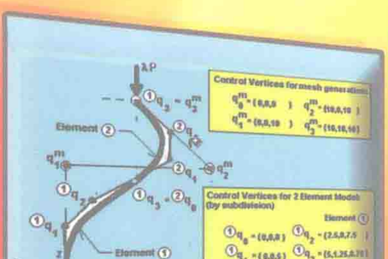
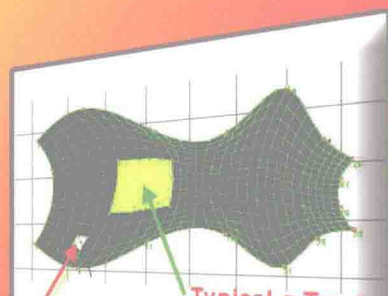
# Computation of Nonlinear Structures

# Extremely Large Elements for Frames, Plates and Shells

Debabrata Ray





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# COMPUTATION OF NONLINEAR STRUCTURES
## EXTREMELY LARGE ELEMENTS FOR FRAMES, PLATES AND SHELLS

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# COMPUTATION OF NONLINEAR STRUCTURES

To the memory of my dad & mom, Provanshu and Sushila Roy, with whose altruistic love it was nurtured

&

the best thing ever happened to me, my wife, Anjana Ray, M.D., with whose infinite patience and eternal support it blossomed

&

the budding analysts and researchers like my sons, Dipanjan Ray, PhD and Shonket Ray, PhD, to whom it is offered

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Finally, however, the faults and mistakes, if any, are entirely mine.

DR
August 31, 2014

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# 1

# Introduction: Background and Motivation

## 1.1 What This Book Is All About

The book introduces linear and nonlinear structural analysis through a combination of of mesh generation, solid mechanics and a new finite element methodology called ***c-type finite element method*** (Ray, 1999, 2003, 2004, 2005, 2007, 2008). The ultimate objective is to present the largest possible (curved) beam, plate and shell elements undergoing extremely large displacement and rotation, and to apply these to solve standard industrial problems. Any finite element method is only as strong as its weakest link. In other words, the book is not just about unification of mesh generation and finite element methodology but it strives to serve as a reference for budding researchers, engineers, analysts, upper division and graduate students and teachers by demonstrating what various interdisciplinary machinery has to be accurately harnessed to devise a solid and conducive theoretical framework upon which to build a robust, reliable and efficient numerical methodology for linear and nonlinear static and dynamic analysis of beams, plates and shells. As indicated, the principal goal of the book is to produce the largest possible arbitrary shaped elements (a) defined and restricted solely by the requirements of geometry, material, loading and support conditions, (b) avoiding computational problems such as locking in the conventional finite element methods and (c) presenting new, accurate and explicit expressions for resolution of the symmetry issue of the tangent operator for beams, plates and shells in areas of extreme nonlinearity. The 'mega-sized' elements may result in substantial cost saving and reduced bookkeeping for the subsequent finite element analysis, and a reduced engineering manpower requirement for the final quality assurance. For example, the explicit algebraic and symmetric expressions of the tangent operator, as presented in the book, are an absolute necessity for computational cost efficiency, especially in repetitive calculations that are commonly associated with nonlinear problems. It must be recognized that the requirements for numerical convergence should be purely adaptive and subservient to the main delineating factors already mentioned. However, this strategy of computer generation of mega-elements of arbitrary shape, as it turns out, takes its toll on the analyst. Firstly, only accurate theoretical formulation can be used for the underlying continuum or solid mechanics principles without unnecessary 'short-circuiting' by proliferation of ad hoc numerical manipulations. Secondly, it demands that the applicable finite element method be devised to successfully accept computer generated elements with arbitrarily distorted shapes, with edges (or faces) consisting of up to truly 3D curved boundaries (or surfaces)

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with natural twist and bend (e.g. for shell elements). Thirdly, for these 'hyperelements' with conformity, the finite element method must be able to accommodate effortlessly and naturally $C^0$, $C^1$ or $C^2$ inter-element continuity on demand.

## 1.2 A Brief Historical Perspective

Every meaningful structural analysis is an exercise in abstraction about a structural system in the real world, so just as with any other natural or man-made phenomenon, the viability and safety of the structural system is intricately associated with the methodology underpinning such abstraction. More specifically, for a structural system, abstractions lie in the geometric modeling of its material body, its relevant support conditions and its imposed loadings, and finally its material properties; we associate the sum total of these abstractions with a structural theory. Moreover, each of these abstractions defines the extent to which a particular structural theory can efficiently and logically predict and control the response of a structural system. To paraphrase Einstein's incisive comment to Heisenberg that led to the latter's discovery of the uncertainty principle, every theory, like a mirror or a horse's blinder, filters and determines what we can see of the real world. Naturally, lest we miss out on important real-world phenomena, as structural analysts, we have to critically evaluate structural theories, propounded in both the distant and the recent past, so that we can be successful in the ultimate goal of our exercise in abstraction, namely, the prediction and control of structural response to external stimuli. Translated into actual methodology for solid bodies, these abstractions reduce to two fundamentally complementary disciplines: solid mechanics and numerical analysis – each determining and harnessing the strength of the other.

### *1.2.1 Operational Mechanics*

For hundreds of years, even before the digital age, the basic theoretical premises of linear and nonlinear solid mechanics involving the study of the deformation of a body, transmission of force through it and the characterization of its material properties, have been well established, and they were perfected during the twentieth century (Eringen, 1962; Green and Zerna, 1968; Ogden, 1997; Malvern, 1969). However, without the computational power of modern computers, the forms of the various equations in solid mechanics, while accurately describing real world problems, were not of much use for finding numerical solutions for material bodies of very complicated geometry and intricate support conditions, subjected to complex loading systems. Thus, before the computer era, solid mechanics had no unifying numerical methodology, and so structural problems were solved on a case-by-case basis using a variety of different analytical methods.

### *1.2.2 Conventional Finite Element Methods*

With the coming of the digital era, numerical methods became much more dominant. Over 60 years ago, an energy-based methodology called the finite element method (Turner, *et al.*, 1956) made its triumphant entry into the realm of numerical structural analysis, its theory having been established earlier (Courant, 1943). Its chief advance was to choose triangular elements with a complete set of basis functions following Pascal's triangle, but soon after, the application of similar basis functions to quadrilateral elements proved to be a poor choice because arbitrarily curtailed and incomplete polynomials resulted in interpolation problems requiring various ad hoc numerical artifacts such as under integration and reduced integration, to rectify numerous locking issues that resulted. Both the conventional finite element methods – h-type with Lagrangian and

Hermite basis functions (Zienkiewicz and Taylor, 2000) and p-type with Legendre basis functions (Szabo and Babuska, 1991), – suffered from these ill-conceived ideas.

While the general theoretical formulation of the displacement finite element method of analysis is based on rigorous variational method, the practice of finite element method is another story – a harrowing experience because, in an ad-hoc manner, it tried in vain to tear apart the fundamental and inalienable dictum of the Rayleigh–Ritz–Galerkin method that says: "... method is a 'package deal', and neither requires nor permits the user to make independent decisions about different parts of the problem" (Strang and Fix, 1973, p. 33). Based on the most common construction mechanism of global basis or Ritz functions from appropriate mapping of the chosen elemental or local basis functions, the ultimate success or failure of a practiced or computer-implemented finite element method essentially depends on the choice of the local basis functions (also known as the elemental shape functions), the adequacy of the mapping functions and the evaluations of the integrals. It is no wonder then that for more than 30 years, the primary focus of all notable fundamental researches in the field of finite element analysis has been on devising the "best" shape functions and associated mapping functions. It is instructive to briefly review the interpolation failure and concomitant patchy, ad hoc remedies with the ill side effects and to present some extremely important conceptual problems that remained unsolved by the conventional finite element methods, insofar as interpolation is concerned.

***h-type Methodology***: Hermitian shape functions, $C^1$ and $C^2$, are used for situations where flexural strain exists such as in the case of beam. The nodes in these cases are what can be considered as multiple nodes; that is, apart from the values, the slopes and/or curvatures are also taken as degrees of freedom at a physical location. The simplest element in the family has two end nodes with two degrees of freedom each for a total of four degrees of freedom. A cubic Hermite is used for this. The inter-element continuity that can be imposed is that the slope or the first derivatives of the functions are continuous, that is, $C^1$ continuity. To speed up convergence where the solution is smooth or has curved flexural elements, one can use a quintic Hermite with curvature or bending moment continuity, that is, $C^2$ continuity. Note that because of the inclusion of the derivative components as degrees of freedom, the interpolation loses its barycentric nature or convexity. The isoparametric elements are obtained following the assumption and procedure that the functional representation of the deformational behavior is employed in representation of the element geometry. In other words, the displacement vector components and the geometry use the same shape functions. This, in turn, forces the conventional element to have as many internal nodes as necessary to make geometry description isoparametric to the displacement function. For example, for a quadratic interpolation function for a truss element, it becomes a three-node element with one internal node. For a conventional isoparametric element, the accuracy increases with the corresponding element size measured by $h$, the diameter of the smallest circle encompassing the element, and, hence, also identified as $h$-type elements. The conventional $h$-type elements in two dimensions are interpolated with polynomials of degree one, two and sometimes three. The main elements, that have been known and used for quite some time, can be categorized into two major groups: Lagrangian and serendipity. These are obtained by taking the tensor product of the one-dimensional Lagrangian elements. The mapping functions are linear. For quadratic elements, say, the Lagrangian tensor product element consists naturally of nine nodes – four corner nodes, four edge nodes and one interior node. For the serendipity element, in this case, the interior node is dropped in favor of incomplete interpolation. The higher-order h-type elements are obtained analogously and may be found in any of the standard texts already mentioned.

***p-type Methodology***: From later developments come the so-called hierarchical elements of increasing polynomial degree $\boldsymbol{p}$, and hence, identified as p-type elements. The main idea behind these elements is that the next improved element of higher degree is achieved by retaining the earlier expressions of all degrees less than the improved one. In other words, to obtain an element of degree $p, p \geq 2$, only higher-order additional terms are introduced to the shape functions of the element of $p-1$ degree. Moreover, these additional shape functions must vanish at old nodes, then since the $p^{\text{th}}$ derivative of an old $(p-1)$ degree shape function always vanishes, this condition is chosen to insert a new node, and finally the new shape function is scaled so that it assumes the value of unity at this generalized displacement node. The combination is not barycentric or convex, and the degrees of freedom consist of both functions and their derivatives as in the mixed formulations. For p-type quadrilateral elements, there is more than one variant of the polynomial spaces. The shape functions are categorized in those spaces into three groups: nodal, side and interior modes. The basic idea behind the shape functions is the same as in one dimension, that is, choosing Legendre polynomials as the primary functions. However, in order to enforce the fundamental property of basis functions, namely that each assumes a value of unity at one node and vanishes at the rest of the nodes, blending functions similar to h-type elements are used.

***Triangular Elements***: The triangular elements, by contrast, have long been described in the barycentric coordinates. The root geometry has been described in three coordinates, suggesting a three-dimensional figure. However, a closer look will reveal that the root element is still planar, that is, a subset of two-dimensional space, $\mathbb{R}^2$, because only two of the coordinates are independent and all three are related; the barycentric coordinates are essentially the area coordinates in two dimensions. In other words, any point in two dimensions can be expressed as convex combinations of the vertex of a triangle containing the point. The coefficients of combinations are the barycentric coordinates of the point. As shown in this book, the three coordinates are proportional to the three smaller triangles generated by joining the point to each vertex of the triangle. All shape functions can be defined on the standard triangle in terms of these barycentric coordinates. For higher-degree h-type triangular elements, edge nodes (e.g. for degree 2) and inside nodes (e.g. for degree three) are included in the element. For the hierarchic p-type elements, for $p = 1, 2$, the shape functions for the triangular elements are exactly the same as those for the h-type elements. The mapping functions are chosen as linear for straight edge triangles. For large curved triangles, mapping functions are isoparametric for $p = 2$. For higher-order triangular elements, mapping functions are developed based on a blending method. For the purpose of critical evaluation to be introduced later, this concludes a brief presentation of the characteristics of the shape functions and the mapping functions for both families, namely the h-type and the p-type, of conventional finite elements.

#### 1.2.2.1 Problems of the Conventional Finite Element Methods

We start by referring to the comment by MacNeal (1994) that the "days of pioneering" or the days of "heuristics, hunches and experimental data to guide . . . design choices" is over, and that we should take design of finite elements ". . . less as an art and more as a science". The noteworthy applied mathematicians such as Strang and Fix appeared to be too apologetic, maybe because the engineers beat the mathematicians and discovered the finite element method; they also appeared to have been temptingly permissive by announcing that the serious violations of the theory are mere "variational crimes": ". . . it [the rule] is broken every day, and for good reason" (MacNeal, 1994). No wonder then that MacNeal is forced to believe that "mathematical rigor falls well short of the goal of converting finite element design into science". In any case, because of these conflicting signals, scores of books keep appearing that simply repeat the same old ("variational crime"-ridden) information relating to the shape functions. The main goal, therefore, will be first to identify the various problems that crept into the mainstream conventional finite element method. However, in order for us to recognize the violations of the conventional finite element method, we need to reiterate the main dictates of the variational theorems: geometry must be accurately interpolated with extreme care for curved boundaries; the trial functions must have adequate derivatives across the element boundaries required by the inter-element continuity requirement; essential boundary conditions must be honored; and the integrals must be computed accurately.

***Power Series as Basis Functions***: One look at the history of the development of finite elements will reveal that primarily the power series made up of monomials of Cartesian position coordinates constituted the basis functions. Naturally, the Pascal triangle had been the guiding post for determining the completeness of the polynomials made up of such monomials. But, innocuous and familiar as these are, the polynomials constructed from these monomials, in their indiscriminate use as basis functions, produced some very distressing phenomena.

***Induced Anisotropy***: Once such commitment to power series has been made – the notion of completeness of the polynomials making up the basis functions became important for both convergence and suitability of the shape functions. Depending on the displacement solution function, convergence of incomplete polynomial basis functions to the solution is not guaranteed. Moreover, the basis functions based on incomplete polynomials can easily induce anisotropy for triangular or tetrahedral elements in the description of, say, the stiffness matrix, that is, the definition has to depend on the orientation of the coordinate axes. Similar loss of symmetry and hence induced anisotropy may result for arbitrary quadrilateral or hexahedral elements.

***Incomplete Strain States***: Incomplete polynomials as basis functions fail to invoke all the internal strain states that may result due to all the possible imposed conditions. For example, a four-node linear conventional quadrilateral element can only represent two internal linear strain states as opposed to required six. In fact, this condition afflicts all serendipity and Lagrange elements in two or three dimensions. Absence of complete strain states gives rise to what is known as ***locking*** which we shall deal with in the chapter on linear application. Based on the preceding discussion, polynomial basis functions made up of monomials do not seem to be a good choice. In fact, p-type hierarchical elements dropped monomials in favor of Legendre polynomials and thus avoided these problems. However, completeness is not the only dominating issue in the choice of the basis functions and mapping functions. We now embark on one such important issue.

***Inter-element Continuity***: As indicated in this book about the theory of Rayleigh–Ritz–Galerkin or finite element method, the trial and test spaces are finite dimensional subspaces of the solution space. The accuracy of the solution increases with the corresponding increase in the dimensionality of the subspaces. Furthermore, the energy in the error is equal to the error in the energy, and the approximated strain energy corresponding to any finite dimensional approximation is always less than that of the actual strain energy associated with exact solution. This, of course, is easy to understand when one considers that the description of an infinite dimensional displacement function by a finite set of basis functions only imposes additional constraints on the system. The additional stiffness, in turn, reduces the displacement and hence the strain energy of the system. The strain energy increases with increase in the number of basis functions. For this to continue to happen, however, the basis functions must satisfy some continuity requirements. For example, for plane stress problems in two dimensions, the trial function, for the Rayleigh–Ritz–Galerkin method, must be such that its first derivative has finite energy, that is, that it must be at least continuous, so that the virtual work expression is meaningful. In general, the trial functions must have finite energy (in the energy norm) in their $m^{\text{th}}$ derivatives when the system-governing differential equation is of order $2m$. In finite element analysis, the geometry is reconstructed as an assemblage of finite elements. The above condition, then, becomes that the basis functions must be of class $C^{m-1}$ across the element boundary. For plane stress analysis, $m = 1$, so the trial functions must be at least $C^0$ across the boundary, whereas for plate analysis, $m = 2$ and the corresponding continuity requirement is $C^1$, that is, the first derivative of the displacement function must be continuous. Any element that violates this requirement is known as the ***non-conforming*** element (Taylor *et al.*, 1976); otherwise, it is ***conforming***. In general, to establish, say, $C^0$ continuity at any edge between any two elements, the shape functions for a node connected to the edge must be identical for the two elements, and the shape functions for the rest of the nodes must vanish at the edge. For example, it can easily be shown that the constant strain, three-node triangle is conforming, and so are triangular or tetrahedral elements with edge nodes, as long as the edges are straight and all edges have same number of nodes. However, for the general four-node quadrilateral element, a simple coordinate transformation by rotation will show that the element is non-conforming. The only quadrilateral element that is conforming is the four-node rectangular element. All other general quadrilateral elements with edge nodes on the straight edges are non-conforming. Finally, if the edges of the elements, triangular or quadrilateral, are curved, the elements are non-conforming. All these elements made up of basis functions by incomplete power series of position coordinates lack the barycentric or convexity property of interpolation, and thus are devoid of invariance under affine – for example, rotational – transformations. In fact, as will be shown later, the problem associated with the general curved conventional elements of today's finite element method is one of interpolation that translates into lack of inter-element continuity in general. Historically speaking, however, these earlier problematic h-type elements found their partial savior in what is known as parametric mapping and rigid body rotation.

***Parametric Mapping and Rigid Body Rotation***: In the theoretical formulation of the isoparametric finite element method, it is assumed that the elemental basis functions are expressed in a root element, and the displacement functions are then mapped isoparametrically with the geometry. For the case where the geometry is straight and can be expressed by a linear root element, the displacement functions could be of higher order, needing conventionally edge or interior nodes; here, the mapping of the root element is known as ***subparametric***. In the reverse situation of curved geometry with simple strain states, the number of basis functions necessary to completely describe the geometry may be higher than that needed for displacement description; this mapping goes by the name of ***superparametric***. For superparametric elements, the representation of rigid body rotation and constant strain states is affected adversely. The history of construction of conventional finite elements with variable nodes appeared to have failed to realize one fundamental mathematical relationship: *the conversion in description of non-parametric functions to parametric description automatically and inescapably fixes all the internal nodes*. The displacement functions or the geometry coordinates, to start with, are non-parametric functions. To generate ***isoparametry*** between them, both needed to be first transformed into parametric entities. This then, by the principle just stated, fixes the intermediate nodes, be it just the edge nodes for one dimension, or the edge, face or internal nodes in their higher dimensional counterpart. Any discussion on distortion of the intermediate nodes is irrelevant and futile. In fact, if complete isoparametry is established, guaranteeing a barycentric or convex combination of the basis functions of the root element, the ensuing finite element, irrespective of any elemental distortion, will be conforming and complete. Short of guaranteed completeness of the polynomials, we would like to have a device by which various conventional elements can be measured for their acceptance. We now engage in discussing one such tool.

***Patch Test***: In earlier days, when terms were being added or dropped for designing new finite elements in ad hoc fashion, a standard numerical test was necessary for evaluating their worth. One such test is the patch test. The idea behind the patch test is simple. A chunk of elements representing a system is plucked out and subjected some known boundary conditions for which the internal solutions are known. It is expected, then, for acceptance of the elements, that the corresponding computed internal quantities should be the same as those known expectations. In other words, if a linear black box (the element) is correct, the output and input must correspond. To make calculations easy, the boundary is taken simple as well, such as square or cube for two and three dimensions, respectively. The constant strain patch test is considered standard for two- dimensional in-plane load conditions, because as the elements reduce sufficiently in size, the state of the strain (or derivative of the displacement) becomes constant in the limit. So, for convergence, the element should be able to represent such a state of strain. For plate bending, this strain becomes the bending curvature. In any case, linear displacement-equivalent forces are applied on the external nodes and solved for internal node displacements. The derivatives of this solution must match the known constant strains. There are several versions of patch tests. A patch test was initially conceived as a numerical experimentation for element validation. For smooth solution problems, the patch test was later shown mathematically to be a necessary and sufficient test for convergence of h-type elements. It appears quite clear to the author that the patch test is, in the main, an interpolation check. However, it serves to detect other failures such as integration or equilibrium failures. Because of the conformity (satisfying equilibrium requirements) and completeness (satisfying interpolation requirements) of the conventional linear isoparametric elements having only corner nodes, generation of constant strain or linear displacement state is extremely easy. Similarly, the conventional variable-node higher-order isoparametric elasticity elements with straight edges or faces with evenly spaced nodes can pass a higher-order patch test. As rectangular elements with evenly spaced edge nodes, the serendipity elements fail the patch test while its counterpart full-noded Lagrangian elements do not. Because of incomplete quadratic terms, isoparametric three- or four-noded bending elements cannot pass the constant curvature or quadratic displacement patch test. For curved elements, triangular (and tetrahedral) or quadrilateral (and hexahedral) with variable nodes arbitrarily placed cannot pass the quadratic displacement or constant patch test as necessary for plate and shell elements. As will be shown in the subsequent chapters, this is clearly an interpolation failure associated with elements with variable nodes under a curved situation. Finally, of course, the linear elements fail to pass the constant curvature patch test. In other words, these elements should not be used to model, say, plate or shell systems.

***Locking Problems***: The discussion on the patch test has been included merely to show the innate interpolation problems that have been afflicting conventional finite elements. More specifically, these translated into other serious numerical maladies known as ***locking***. We will therefore now look at the phenomenon of locking only in conventional rectangular elements. Other elements such as triangular elements experience the same locking phenomena, so we will dscribe the remedial methods that have been tried, but not altogether successfully. The futility crept in because none of these so-called tricks ever tried to correct the innate interpolation problems. In fact, these remedies, which include reduced and selective integration, incompatible modes and so on, do not appear to be numerically robust in the classical sense of consistency, convergence and stability. As a result, we are burdened with the additional problem of spurious modes introduced by reduced or selective integration. Simple solutions based on exact interpolation theory that allow the reader to appreciate the strength of the new c-type finite element method are presented in Chapter 9 about linear applications.

***Shear and Membrane Locking and Shape Dependence***: The major debacle of interpolation failure is locking, that is, elements showing unacceptably high stiffness, almost bordering on rigidity, in certain displacement states. In other words, the low order or variable node elements are failing to respond flexibly enough for or failing to recognize as it should, certain imposed conditions. Moreover, it was recognized that the severity of this phenomenon of locking depends on the shape of the element. Finally, the shape may be boiled down to a specific parameter that allegedly perpetrates such locking.

Now let us briefly examine the conventional state-of-the-art remedies.

***Reduced/Selective Integration, Bubble Mode and Drilling Freedom***: The linear four-node element failed to interpolate the quadratic term, and thus experienced dilatation locking.