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ISOPERIMETRIC INEQUALITIES

Differential Geometric and Analytic Perspectives

等周不等式

ISAAC CHAVEL

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This introduction treats the classical isoperimetric inequality in Euclidean space and contrasting rough inequalities in noncompact Riemannian manifolds. In Euclidean space the emphasis is on quantitative precision for very general domains, and in Riemannian manifolds the emphasis is on qualitative features of the inequality that provide insight into the coarse geometry at infinity of Riemannian manifolds.

The treatment in Euclidean space features a number of proofs of the classical inequality in increasing generality, providing in the process a transition from the methods of classical differential geometry to those of modern geometric measure theory; and the treatment in Riemannian manifolds features discretization techniques and applications to upper bounds of large time heat diffusion in Riemannian manifolds.

The result is an introduction to the rich tapestry of ideas and techniques of isoperimetric inequalities, a subject that has beginnings in classical antiquity and that continues to inspire fresh ideas in geometry and analysis to this very day – and beyond.

Isaac Chavel is Professor of Mathematics at the City College of The City University of New York. He received his Ph.D. in mathematics from Yeshiva University under the direction of Professor Harry E. Rauch. He has published in international journals in the areas of differential geometry and partial differential equations, especially the Laplace and heat operators on Riemannian manifolds. His other books include *Eigenvalues in Riemannian Geometry* and *Riemannian Geometry: A Modern Introduction*.

He has been teaching at the City College of The City University of New York since 1970, and he has been a member of the doctoral program of The City University of New York since 1976. He is a member of the American Mathematical Society.

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Preface

This book discusses two venues of the isoperimetric inequality: (i) the sharp inequality in Euclidean space, with characterization of equality, and (ii) isoperimetric inequalities in Riemannian manifolds, where precise inequalities are unavailable but rough inequalities nevertheless yield qualitative global geometric information about the manifolds.

In Euclidean space, a variety of proofs are presented, each slightly more ambitious in its application to domains with irregular boundaries. One could easily go directly to the final definitive theorem and proof with little ado, but then one would miss the extraordinary wealth of approaches that exist to study the isoperimetric problem. An idea of the overwhelming variety of attack on this problem can be quickly gleaned from the fundamental treatise of Burago and Zalgaller (1988); and I have attempted on the one hand to capture some of that variety, and on the other hand to find a more leisurely studied approach that covers less material but with more detail.

In Riemannian manifolds, the treatment is guided by two motifs: (a) the dichotomy between the local Euclidean character of all Riemannian manifolds and the global geometric properties of Riemannian manifolds, this dichotomy pervading the study of nearly all differential geometry, and (b) the discretization of Riemannian manifolds possessing bounded geometry (some version of local uniformity). The dichotomy between local and global is expressed, in our context here, as the study of properties of Riemannian manifolds that remain invariant under the replacement of a compact subset of the manifold with another of different geometry and topology, as long as the new one fits smoothly in the manifold across the boundary of the deletion of the original compact subset. Thus, we do not seek fine results, in that we study coarse robust invariants that highlight the "geometry at infinity" of the manifold. Our choice of isoperimetric constants will even be invariant with respect to the discretization of the Riemannian manifold. The robust character of these new isoperimetric constants will then allow us to use this discretization to show how the geometry at infinity influences large time heat diffusion on Riemannian manifolds.

Regrettably, there is hardly any discussion of isoperimetric inequalities on compact Riemannian manifolds. That would fill a book – quite different from this one – all by itself.

* * *

A summary of the chapters goes as follows:

Chapter I starts with posing the isoperimetric problem in Euclidean space and gives some elementary arguments toward its solution in the Euclidean plane. These arguments are essentially a warm-up. They are followed by a summary of background definitions and results to be used later in the book. Thus the discussion of the isoperimetric problem, proper, begins in Chapter II.

Chapter II starts with uniqueness theory, under the assumption that the boundary of the solution domain is C^2. We first show that, if a domain Ω with C^2 boundary is a solution to the isoperimetric problem for domains with C^2 boundaries, Ω must be an open disk. Then we strengthen the result a bit – we show that if a domain is but an extremal for isoperimetric problems, then it must be a disk. Then we consider the existence of a solution to the isoperimetric problem for domains with C^1 boundaries. We give M. Gromov's argument that for such domains the disk constitutes a solution to the isoperimetric problem. But only if one restricts oneself to convex domains with C^1 boundaries does his argument imply that the disk is the unique solution.

Chapter III is the heart of the first half of the book. It expands the isoperimetric problem in that it considers all compacta and assigns the Minkowski area to each compact subset of Euclidean space to describe the size of the boundary. In this setting, using the Blaschke selection theorem and Steiner symmetrization, one shows that the closed disk constitutes a solution to the isoperimetric problem. Since the Minkowski area of a compact domain with C^1 boundary is the same as the differential geometric area of the boundary, the result extends the solution of the isoperimetric problem from the C^1 category to compacta. Moreover, one can use the traditional calculations to show that the disk is the unique solution to the isoperimetric problem in the C^1 category. But uniqueness in the more general collection of compacta is too difficult for such elementary arguments.

Then, in Chapter III, we recapture Steiner's original intuition that successive symmetrizations could be applied to any compact set to ultimately have it converge to a closed disk – in the topology of the Hausdorff metric on compact sets. We use this last argument to prove the isoperimetric inequality for compacta with finite perimeter. The perimeter, as a measure of the area of the boundary, seems to be an optimal general setting, since one can not only prove the isoperimetric inequality for compacta with finite perimeter, but can also characterize the case of equality.

In Chapter IV we introduce Hausdorff measure for subsets of Euclidean space and develop the story sufficiently far to prove that the perimeter of a Lipschitz domain in n-dimensional Euclidean space equals the $(n-1)$-dimensional Hausdorff measure of its boundary. The proof involves the *area formula*, for which we include a proof.

Chapter V begins a new view of isoperimetric inequalities, namely, rough inequalities in a Riemannian manifold. The goal of Chapters V–VIII is to show how these geometric isoperimetric inequalities influence the qualitative rate of decay, with respect to time, of heat diffusion in Riemannian manifolds.

In Chapter V we summarize the basic notions and results concerning isoperimetric inequalities in Riemannian manifolds, and in Chapter VI we give their implications for analytic Sobolev inequalities on Riemannian manifolds. Chapter V consists, almost entirely, of a summary of results from my *Riemannian Geometry: A Modern Introduction*, and I have included just those proofs that seemed to be important to the discussion here. The discussion of Sobolev inequalities in Chapter VI has received extensive treatment in other books, but our interest is restricted to those inequalities required for subsequent applications. Moreover, we have also treated the relation of Sobolev inequalities on Riemannian manifolds and their discretizations, one to the other. To my knowledge, this has yet to be treated systematically in book form.

Chapter VII introduces the Laplacian and the heat operator on Riemannian manifolds and is devoted to setting the stage for the "main problem" in large time heat diffusion; its formulation and solution are presented in Chapter VIII. The book ends with an introduction to the new arguments of A.A. Grigor'yan, the full possibilities of which have only begun to be realized.

* * *

I have attempted to strike the right balance between merely summarizing background material (of which there is quite a bit) and developing preparatory arguments in the text. Also, although I have summarized the necessary basic definitions and results from Riemannian geometry at the beginning of Chapter V, I occasionally require some of that material in earlier chapters, and I use it as though the reader already knows it. This seems the lesser of two evils, the other evil being to disrupt the flow of the arguments in the first half of the book for an excursus that would have to be repeated in its proper context later. Most of that material is quite elementary and standard, so it should not cause any major problems.

In order to clarify somewhat the relation between material quoted and material presented with proofs, I have referred to every result that either is an exercise or that relies on a treatment outside this book as a proposition, and every result proven in the book as a theorem. This is admittedly quite artificial and obviously gives rise to some strange effects, in that the titles *proposition* and *theorem* are often (if not usually) used to indicate the relative significance of the results discussed. That is not the case here.

There are bibliographic notes at the end of each chapter. They are intended to give the reader some guidance to the background material, and to give but an introduction to a definitive study of the literature.

It is a pleasure to thank the many people with whom I have been associated in the study of geometry since I first came to the City College of CUNY in 1970: first and foremost, Edgar A. Feldman and the other geometers of the City University of New York – J. Dodziuk, L. Karp, B. Randol, R. Sacksteder, and J. Velling. Also, I have benefited through the years from the friendship and mathematics of I. Benjamini, M. van den Berg, P. Buser, E. B. Davies, J. Eels, D. Elworthy, A. A. Grigor'yan, E. Hsu, W. S. Kendall, F. Morgan, R. Osserman, M. Pinsky and D. Sullivan. But, as is well known, any mistakes herein are all mine.

The isoperimetric problem has been a source of mathematical ideas and techniques since its formulation in classical antiquity, and it is still alive and well in its ability to both capture and nourish the mathematical imagination. This book only covers a small portion of the subject; nonetheless, I hope the presentation gives expression to some of its beauty and inspiration.

Contents

I
Introduction

In this chapter we introduce the subject. We describe the classical isoperimetric problem in Euclidean space of all dimensions, and give some elementary arguments that work in the plane. Only one approach will carry over to higher dimensions, namely, the necessary condition established by classical calculus of variations, that a domain with C^2 boundary provides a solution to the isoperimetric problem only if it is a disk. Then we give a recent proof of the isoperimetric inequality in the plane by P. Topping (which does not include a characterization of equality), and the classical argument of A. Hurwitz to prove the isoperimetric inequality using Fourier series. This is followed by a symmetry and convexity argument in the plane for very general boundaries that proves the isoperimetric inequality, *if* one assumes in advance that the *isoperimetric functional* $D \mapsto L^2(\partial D)/A(D)$ has a minimizer. (So this is a weak version – if the isoperimetric problem has a solution, then the disk is also a solution.) Finally, we present the background necessary for what follows later in our general discussion, valid for all dimensions. The subsections of §I.3 include a proof of H. Rademacher's theorem on the almost everywhere differentiability of Lipschitz functions, and a proof of the general co-area formula for C^1 mappings of Riemannian manifolds. We obtain the usual co-area formula, as well as an easy consequence: Cauchy's formula for the area of the boundary of a convex subset of $\mathbb{R}^n$ with C^1 boundary.

I.1 The Isoperimetric Problem

Given any bounded domain on the real line (that is, an open interval), the discrete measure of its boundary (the endpoints of the interval) is 2. And given any bounded open subset of the line, the discrete measure of its boundary is greater than or equal to 2, with equality if and only if the open set consists of one open interval. This is the statement of the isoperimetric inequality on the line.

In the plane, one has three common formulations of the *isoperimetric problem*:

1. Consider all bounded domains in $\mathbb{R}^2$ with fixed given perimeter, length of the boundary (that is, all domains under consideration are *isoperimetric*). Find the domain that contains the greatest area. The answer, of course, will be the disk. Note that the specific value of the perimeter in question is of no interest, because all domains of perimeter L_1 are mapped by a similarity of $\mathbb{R}^2$ to all domains with perimeter L_2 for any given values of L_1, L_2, and the image under the similarity of an area maximizer for L_1 is an area maximizer for L_2.
2. One insists on a common area of all bounded domains under consideration, and asks how to minimize the perimeter.
3. Lastly, one expresses the problem as an analytic inequality, namely, since we know exactly the values of the area of the disk and the length of its boundary, the isoperimetric problem is then expressed as proving the *isoperimetric inequality*

   (I.1.1) $$L^2 \geq 4\pi A,$$

   where A denotes the area of the domain under consideration, and L denotes the length of its boundary. The inequality is extremely convenient, in that it remains invariant under similarities of $\mathbb{R}^2$, and one has equality if the domain is a disk. One wishes to show that the inequality is always true, with equality if and only if the domain is a disk.

One can consider the above for any $\mathbb{R}^n$, $n \geq 2$. The proposed analytic isoperimetric inequality then becomes

(I.1.2) $$\frac{A(\partial\Omega)}{V(\Omega)^{1-1/n}} \geq \frac{A(\mathbb{S}^{n-1})}{V(\mathbb{B}^n)^{1-1/n}},$$

where Ω is any bounded domain in $\mathbb{R}^n$ and $\partial\Omega$ its boundary, V denotes n-measure and A denotes $(n-1)$-measure, $\mathbb{B}^n$ is the unit disk in $\mathbb{R}^n$, and $\mathbb{S}^{n-1}$ the unit sphere in $\mathbb{R}^n$. We let ω_n denote the n-dimensional volume of $\mathbb{B}^n$ and $\mathbf{c}_{n-1}$ the $(n-1)$-dimensional surface area of $\mathbb{S}^{n-1}$. It is standard that

(I.1.3) $$\mathbf{c}_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}, \qquad \omega_n = \frac{\mathbf{c}_{n-1}}{n},$$

where $\Gamma(x)$ denotes the classical gamma function; and (I.1.2) now reads as

(I.1.4) $$\frac{A(\partial\Omega)}{V(\Omega)^{1-1/n}} \geq n{\omega_n}^{1/n}.$$

One wants to prove the inequality and to show that equality is achieved if and only if Ω is an n-disk. Note that for $n = 2$ we took in (I.1.4) the square root of (I.1.1).

Remark I.1.1 Throughout the book, *domain* will refer to a connected open set. In general, we consider the isoperimetric problem for relatively compact domains when we are working in the differential geometric setting (Chapters I, II, V–VIII). Therefore, the disks that realize the solution in $\mathbb{R}^n$ are open. In Chapters III and IV, where we work in a more general setting, the isoperimetric problem is considered for compacta. In that setting the disks that realize the solution in $\mathbb{R}^n$ are closed.

Remark I.1.2 We have restricted the isoperimetric problem to domains in $\mathbb{R}^n$; but if we could solve this problem, then the isoperimetric problem for open sets consisting of finitely many bounded domains would easily follow from the solution for single domains. Indeed, assume one has the inequality (I.1.2) for domains in $\mathbb{R}^n$. If

$$\Omega = \Omega_1 \cup \Omega_2 \cup \cdots,$$

where each Ω_j is a relatively compact domain in $\mathbb{R}^n$ such that

$$\mathrm{cl}\,\Omega_j \cap \mathrm{cl}\,\Omega_k = \emptyset \qquad \forall\, j \neq k$$

(cl denotes the closure), then Minkowski's inequality implies

$$\begin{aligned} V(\Omega)^{1-1/n} &\leq \sum_j V(\Omega_j)^{1-1/n} \leq \frac{1}{n\omega_{\mathrm{n}}{}^{1/n}} \sum_j A(\partial\Omega_j) \\ &= \frac{1}{n\omega_{\mathrm{n}}{}^{1/n}} A(\partial\Omega). \end{aligned} \tag{I.1.5}$$

So the inequality extends to the union of domains. Note that equality implies that Ω is a domain.

Remark I.1.3 Note that for any domain Ω in $\mathbb{R}^n$, its volume is the n-dimensional Lebesgue measure, and if $\partial\Omega$ is C^1 then the area of $\partial\Omega$ is given by the standard differential geometric surface area of a smooth hypersurface in $\mathbb{R}^n$. However, if $\partial\Omega$ is not smooth, then one must propose an area functional defined on a collection of domains such that the area functional will give a working definition of the area of the boundaries of the domains. Besides a number of natural properties [see the discussions in Burago and Zalgaller (1988)], one requires that the new definition agree with the differential geometric one when applied to a domain with smooth boundary. Then, with this new collection of domains and definition of the area of their boundaries, one wishes to prove the isoperimeric inequality. Also, one wishes to characterize the case of equality in each of these settings.

Remark I.1.4 As soon as one expands the problem to the model spaces of constant sectional curvature, that is, to spheres and hyperbolic spaces, one has no self-similarities of the Riemannian spaces in question. And if the disks on the right hand side of (I.1.2) are to have radius r, then the right hand side of the inequality in (I.1.2) is no longer independent of the value of r. Nonetheless, one still has the isoperimetric inequality in the sense that all domains in question with the same n-volume have the $(n-1)$-area of their boundaries minimized by disks. For $n = 2$, the analytic formulation reads as follows: If $M = \mathbb{M}^2_\kappa$, the model space with constant curvature κ, then the isoperimetric inequality becomes

$$L^2 \geq 4\pi A - \kappa A^2, \tag{I.1.6}$$

with equality if and only if the domain in question is a disk. Of course, one can still consider the isoperimetric problem, whether or not it is to be expressed as an inequality, in the first or second formulation above.

Similarly, one can extend the isoperimetric problem and associated inequalities to surfaces, or, more generally, to Riemannian manifolds. We shall consider such inequalities in Chapter V.

Remark I.1.5 Finally, one can consider a *Bonnesen inequality*. In $\mathbb{R}^2$, such an inequality is of the form

$$L^2 - 4\pi A \geq B \geq 0,$$

where B is a nonnegative geometric quantity associated with the domain that vanishes if and only if the domain is a disk.

I.2 The Isoperimetric Inequality in the Plane

For any C^2 path $\omega : (\alpha, \beta) \to \mathbb{R}^2$ in the plane, the velocity vector field of ω is given by its derivative ω', and acceleration vector field by ω''. We assume that ω is an immersion, that is, ω' never vanishes. The infinitesimal element of arc length ds is given by

$$ds = |\omega'(t)|\, dt.$$

Given any $t_0 \in (\alpha, \beta)$, the *arc length function* of ω based at t_0 is given by

$$s(t) = \int_{t_0}^{t} |\omega'(\tau)|\, d\tau.$$

Let

$$\mathbf{T}(t) = \frac{\omega'(t)}{|\omega'(t)|}$$

denote the unit tangent vector field along ω,

$$\iota : \mathbb{R}^2 \to \mathbb{R}^2$$

the rotation of $\mathbb{R}^2$ by $\pi/2$ radians, and

$$\mathbf{N} = \iota\,\mathbf{T}$$

the oriented unit normal vector field along ω. Then one defines the *curvature* κ of ω by

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N} \tag{I.2.1}$$

(indeed, since $\mathbf{T}$ is a unit vector field, its derivative must be perpendicular to itself). Then the formula for the curvature, relative to the original path, is given by

$$\kappa = \frac{d\mathbf{T}}{ds}\cdot\mathbf{N} = \frac{\omega''\cdot\iota\omega'}{|\omega'|^3}.$$

One can easily show that

$$\frac{d\mathbf{N}}{ds} = -\kappa\mathbf{T}. \tag{I.2.2}$$

The equations (I.2.1) and (I.2.2) are referred to as the *Frenet formulae.*

One can prove, from (I.2.1), that if the curvature κ is constant, then ω is an arc on a circle (if not the complete circle).

I.2.1 Uniqueness for Smooth Boundaries

As a warm-up, we give the argument from classical calculus of variations. Given the area A, let D vary over relatively compact domains in the plane of area A, with C^1 boundary, and suppose the domain Ω, $\partial\Omega \in C^2$, realizes the minimal boundary length among all such domains D. We claim that Ω is a disk.

Proof Since Ω is relatively compact in $\mathbb{R}^2$, there exists a simply connected domain Ω_0 such that

$$\Omega = \Omega_0 \setminus \{\text{finite disjoint union of closed topological disks}\}.$$

We claim that since Ω is a minimizer, then $\Omega_0 = \Omega$. If not, we may add the topological disks to Ω, which will increase the area of the domain and decrease