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Perturbation Methods in Applied Mathematics

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Springer-Verlag
New York Heidelberg Berlin

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AMS Subject Classification (1980): 34D10, 34E10, 35B20, 35B25

Library of Congress Cataloging in Publication Data

Kevorkian, J
Perturbation methods in applied mathematics.

(Applied mathematical sciences; v. 34)
A revised and updated version of J. D. Cole's
Perturbation methods in applied mathematics, 1968.
Bibliography: p.
Includes index.
1. Differential equations—Numerical solutions.
2. Differential equations—Asymptotic theory.
3. Perturbation (Mathematics) I. Cole, Julian D.,
joint author. II. Title. III. Series.
QA1.A647 vol. 31 [Q371] 510s [515.3'5] 80-16796

With 79 figures.

Printed in the United States of America.

9 8 7 6 5 4 3 2 1

ISBN 0-387-90507-3 Springer-Verlag New York
ISBN 3-540-90507-3 Springer-Verlag Berlin Heidelberg

Preface

This book is a revised and updated version, including a substantial portion of new material, of J. D. Cole's text *Perturbation Methods in Applied Mathematics*, Ginn–Blaisdell, 1968. We present the material at a level which assumes some familiarity with the basics of ordinary and partial differential equations. Some of the more advanced ideas are reviewed as needed; therefore this book can serve as a text in either an advanced undergraduate course or a graduate level course on the subject.

The applied mathematician, attempting to understand or solve a physical problem, very often uses a perturbation procedure. In doing this, he usually draws on a backlog of experience gained from the solution of similar examples rather than on some general theory of perturbations. The aim of this book is to survey these pertubation methods, especially in connection with differential equations, in order to illustrate certain general features common to many examples. The basic ideas, however, are also applicable to integral equations, integrodifferential equations, and even to difference equations.

In essence, a perturbation procedure consists of constructing the solution for a problem involving a small parameter ε, either in the differential equation or the boundary conditions or both, when the solution for the limiting case $\varepsilon = 0$ is known. The main mathematical tool used is asymptotic expansion with respect to a suitable asymptotic sequence of functions of ε.

In a regular perturbation problem a straightforward procedure leads to an approximate representation of the solution. The accuracy of this approximation does not depend on the value of the independent variable and gets better for smaller values of ε. We will not discuss this type of problem here as it is well covered in other texts. For example, the problem of calculating the perturbed eigenvalues and eigenfunctions of a self adjoint differential operator is a regular perturbation problem discussed in most texts on differential equations.

Rather, this book concentrates on singular perturbation problems which are very common in physical applications and which require special techniques. Such singular perturbation problems may be divided into two broad categories: layer-type problems and cumulative perturbation problems.

In a layer-type problem the small parameter multiplies a term in the differential equation which becomes large in a thin layer near a boundary (e.g., a boundary-layer) or in the interior (e.g., a shock-layer). Often, but not always, this is the highest derivative in the differential equation and the $\varepsilon = 0$ approximation is therefore governed by a lower order equation which cannot satisfy all the initial or boundary conditions prescribed. In a cumulative perturbation problem the small parameter multiplies a term which never becomes large. However, its cumulative effect becomes important for large values of the independent variable. In some applications both categories occur simultaneously and require the combined use of the two principal techniques we study in this book.

This book is written very much from the point of view of the applied mathematician; much less attention is paid to mathematical rigor than to rooting out the underlying ideas, using all means at our disposal. In particular, physical reasoning is often used as an aid to understanding a problem and to formulating the appropriate approximation procedure.

The first chapter contains some background on asymptotic expansions. The more advanced techniques in asymptotics such as the methods of steepest descents and stationary phase are not covered as there are excellent modern texts including these techniques which, strictly speaking, are not perturbation techniques. In addition, we introduce in this chapter the basic ideas of limit process expansions, matching asymptotic expansions, and general asymptotic expansions.

Chapter 2 gives a deeper exposition of limit process expansions through a sequence of examples for ordinary differential equations. Chapter 3 is devoted to cumulative perturbation problems using the so-called multiple variable expansion procedure. Applications to nonlinear oscillations, flight mechanics and orbital mechanics are discussed in detail followed by a survey of other techniques which can be used for this class of problems.

In Chapter 4 we apply the procedures of the preceding chapters to partial differential equations, presenting numerous physical examples. Finally, the last chapter deals with a typical use of asymptotic expansions, the construction of approximate equations; simplified models such as linearized and transonic aerodynamics, and shallow water theory are derived from more exact equations by means of asymptotic expansions. In this way the full meaning of laws of similitude becomes evident.

The basic ideas used in this book are, as is usual in scientific work, the ideas of many people. In writing the text, no particular attempt has been made to cite the original authors or to have a complete list of references and bibliography. Rather, we have tried to present the "state of the art" in a systematic manner starting from elementary applications and progressing gradually to areas of current research.

For a deeper treatment of the fundamental ideas of layer-type expansions and related problems the reader is referred to the forthcoming book by P. A. Lagerstrom and J. Boa of Caltech.

To a great extent perturbation methods were pioneered by workers in fluid mechanics and these traditional areas are given full coverage. Applications in celestial mechanics, nonlinear oscillations, mathematical biology, wave propagation, and other areas have also been successfully explored since the publication of J. D. Cole's 1968 text. Examples from these more recent areas of application are also covered.

We believe that this book contains a unified account of perturbation theory as it is understood and widely used today.

Fall 1980

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Chapter 1

Introduction

1.1 Ordering

We will use the conventional order symbols as a mathematical measure of the relative order of magnitude of various quantities. Although generalizations are straightforward, we need only be concerned with scalar functions of real variables. In the definitions which follow ϕ, ψ, etc. are scalar functions of the variable x (which may be a vector) and the scalar parameter ε. The variable x ranges over some domain D and ε belongs to some interval I.

Large O

Let x be fixed. We say $\phi = O(\psi)$ in I if there exists a $k(x)$ such that $|\phi| \leq k(x)|\psi|$ for all ε in I. Similarly, if ε_0 is a limit point in I we say that $\phi = O(\psi)$ as $\varepsilon \to \varepsilon_0$ if there exists a $k(x)$ and a neighborhood N of ε_0 such that $|\phi| \leq k(x)|\psi|$ for all ε in the intersection of N with I.

We note that if ψ does not vanish in I then the inequality in the above two definitions simply reduces to the statement that ϕ/ψ is bounded.

Small o

Again with x fixed, we say $\phi = o(\psi)$ as $\varepsilon \to \varepsilon_0$ if given any $\delta(x) > 0$, there exists a neighborhood N_δ of ε_0 such that $|\phi| \leq \delta(x)|\psi|$ for all ε in N_δ. Here also the definition simplifies to the statement that $(\phi/\psi) \to 0$ if $\psi \neq 0$ in I. Often, $\phi \ll \psi$ is used as an equivalent notation.

Uniformity

As indicated in the above definitions the quantities k, δ and the neighborhoods N, N_δ will, in general, depend on the value of x. If, however k, δ, N, N_δ

can be found independently of the value of x we say that the order relations hold *uniformly in D*. To illustrate these ideas consider the following examples. In all cases x will be a real variable, the domain D will be the half-open unit interval $0 < x \leq 1$ and I will be the half-open interval $0 < \varepsilon \leq \mu < 1$ with $\varepsilon_0 = 0$.

(i) $$x + \varepsilon = O(1) \quad \text{in } I, \text{ uniformly in } D. \tag{1.1.1}$$

(ii) $$\log(\sin \varepsilon x) = O(\log 2\varepsilon x/\pi) \text{ in } I, \text{ uniformly in } D. \tag{1.1.2}$$

This follows from the fact that $0 < 2z/\pi < \sin z$ for all $0 < z < 1$. Since $0 < \varepsilon x < 1$ always, the inequality in the definition holds with $k = 1$.

(iii) $$\frac{1}{x + \varepsilon} = O(1) \quad \text{in } I. \tag{1.1.3}$$

The statement is true because $1/(x + \varepsilon) < 1/x$ for any given x and all ε in I; thus $k(x) = 1/x$. Now, it is clear that the statement (1.1.3) is not uniformly valid in D because there is no finite constant k for which the required inequality holds for all x in D so long as x is allowed to approach the origin.

For similar reasons the statement $\varepsilon/x(1 - x) = O(\varepsilon)$ in I is not uniformly valid in $0 < x < 1$.

(iv) $$\varepsilon^\alpha = O(\varepsilon^\beta) \quad \text{in } I \text{ for any } \alpha \geq \beta. \tag{1.1.4}$$

This result is trivially true since $\varepsilon^{\alpha - \beta}$ is bounded. In fact, it tends to zero for $\alpha > \beta$, and this is a reminder that the O symbol does not connote equality of order of magnitude but only provides a *one-sided* bound.

(v) $$\sin \frac{x}{\varepsilon} = O(x) \quad \text{as } \varepsilon \to 0. \tag{1.1.5}$$

Here, even though the limit as $\varepsilon \to 0$ of $\sin(x/\varepsilon)$ does not exist for any $x \neq 0$, it is clear that $|\sin x/\varepsilon| \leq 1$ for any x in D. Therefore, (1.1.5) is true with $k(x) = 1/x$ and the statement is not uniformly valid. However, the statement $\sin x/\varepsilon = O(1)$ as $\varepsilon \to 0$ is uniformly valid in D.

(vi) $$\varepsilon^\alpha = o(\varepsilon^\beta) \quad \text{as } \varepsilon \to 0 \text{ if } \alpha > \beta. \tag{1.1.6}$$

(vii) $$\varepsilon^\alpha \log \varepsilon = o(1) \quad \text{as } \varepsilon \to 0 \text{ for any } \alpha > 0. \tag{1.1.7}$$

(viii) $$e^{-x/\varepsilon} = o(\varepsilon^\beta) \quad \text{as } \varepsilon \to 0 \text{ for any } \beta \geq 0 \text{ if } x > 0. \tag{1.1.8}$$

Clearly, the statement (1.1.8) is not uniformly valid, even in the half-open interval $0 < x \leq 1$, and if $x = 0$ the statement is false.

Various operations such as addition, multiplication and integration can be performed with the order relations. In general, differentiation of order relations with respect to ε or x is not permissible. For these and further results the reader may consult Reference 1.1.1.

Reference

1.1.1 A. Erdelyi, *Asymptotic Expansions*, Dover Publications, New York, 1956.

1.2 Asymptotic Sequences and Expansions

Consider a sequence $\{\phi_n(\varepsilon)\}$ $n = 1, 2, \ldots$ of functions of ε. Such a sequence is called an asymptotic sequence if

$$\phi_{n+1}(\varepsilon) = o(\phi_n(\varepsilon)) \quad \text{as } \varepsilon \to \varepsilon_0 \tag{1.2.1}$$

for each $n = 1, 2, \ldots$

If the sequence is infinite and $\phi_{n+1} = o(\phi_n)$ uniformly in n (i.e., the choice of δ and N_δ in the definition given in Section 1.1 does not depend on n) the sequence is said to be uniform in n. Similarly if the ϕ_n also depend on a variable x one can have uniformity with respect to x in some domain D. Some examples of asymptotic sequences are

$$\phi_n(\varepsilon) = (\varepsilon - \varepsilon_0)^n, \quad \text{as } \varepsilon \to \varepsilon_0 \tag{1.2.2}$$

$$\phi_n(\varepsilon) = e^{\varepsilon}\varepsilon^{-\lambda_n}, \quad \text{as } \varepsilon \to \infty,\ \lambda_{n+1} > \lambda_n \tag{1.2.3}$$

$$\begin{aligned} &\phi_0 = \log \varepsilon, \quad \phi_1 = 1, \quad \phi_2 = \varepsilon \log \varepsilon, \quad \phi_3 = \varepsilon \\ &\phi_4 = \varepsilon^2 \log^2 \varepsilon, \quad \phi_5 = \varepsilon^2 \log \varepsilon, \quad \phi_6 = \varepsilon^2 \ldots, \\ &\qquad\qquad\qquad\qquad \text{as } \varepsilon \to 0. \end{aligned} \tag{1.2.4}$$

Here again various operations, such as multiplication of two sequences or integration can be used to generate a new sequence. Differentiation with respect to ε may not lead to a new asymptotic sequence. For more details the reader may use Reference 1.1.1.

A sum of terms of the form $\sum_{n=1}^{N} a_n(x)\phi_n(\varepsilon)$ is called an asymptotic expansion of the function $f(x, \varepsilon)$ to N terms (N may be infinite) as $\varepsilon \to \varepsilon_0$ with respect to the sequence $\{\phi_n(\varepsilon)\}$ if

$$f(x, \varepsilon) - \sum_{n=1}^{M} a_n(x)\phi_n(\varepsilon) = o(\phi_M) \quad \text{as } \varepsilon \to \varepsilon_0 \tag{1.2.5}$$

for each $M = 1, 2, \ldots, N$.

If $N = \infty$, the following notation is generally used

$$f(x, \varepsilon) \sim \sum_{n=1}^{\infty} a_n(x)\phi_n(\varepsilon) \quad \text{as } \varepsilon \to \varepsilon_0. \tag{1.2.6}$$

Clearly, an equivalent definition for an asymptotic expansion is that

$$f(x, \varepsilon) - \sum_{n=1}^{M-1} a_n(x)\phi_n(\varepsilon) = O(\phi_M) \quad \text{as } \varepsilon \to \varepsilon_0 \tag{1.2.7}$$

for each $M = 2, \ldots, N$.

An asymptotic expansion is said to be uniformly valid in some domain D in x if the order relations in (1.2.5) or (1.2.7) hold uniformly.

Given a function $f(x, \varepsilon)$ and an asymptotic sequence $\{\phi_n(\varepsilon)\}$, one can uniquely calculate each of the $a_n(x)$ defining the asymptotic expansion of $f(x, \varepsilon)$ by repeated application of the definition (1.2.5). Thus,

$$a_1(x) = \lim_{\varepsilon \to \varepsilon_0} \frac{f(x, \varepsilon)}{\phi_1(\varepsilon)} \tag{1.2.8a}$$

$$a_2(x) = \lim_{\varepsilon \to \varepsilon_0} \frac{f(x, \varepsilon) - a_1(x)\phi_1(\varepsilon)}{\phi_2(\varepsilon)} \tag{1.2.8b}$$

$$a_k(x) = \lim_{\varepsilon \to \varepsilon_0} \frac{f(x, \varepsilon) - \sum_{n=1}^{k-1} a_n(x)\phi_n(\varepsilon)}{\phi_k(\varepsilon)}. \tag{1.2.8c}$$

For example $f(x, \varepsilon) = (x + \varepsilon)^{-1/2}$ has the expansion

$$(x + \varepsilon)^{-1/2} \sim \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{n-1}(n-1)!} \prod_{k=1}^{n} |2k - 3| \frac{\varepsilon^{n-1}}{x^{(2n-1)/2}} \tag{1.2.9}$$

as $\varepsilon \to 0$, with respect to the sequence $\{\varepsilon^{n-1}\}$. This is also the Taylor series expansion of $(x + \varepsilon)^{-1/2}$ near $\varepsilon = 0$ and is convergent for $\varepsilon < |x|$. Note also that the expansion (1.2.9) is not uniformly valid in any domain in x for which $x = 0$ is a limit point.

A less trivial situation occurs if $f(x, \varepsilon)$ is defined by an integral representation. Consider, for example, the Error function defined by

$$\operatorname{erf} \varepsilon = 1 - \frac{2}{\sqrt{\pi}} \int_{\varepsilon}^{\infty} e^{-t^2}\, dt \tag{1.2.10a}$$

which by setting $t^2 = \tau$ can also be written as

$$\operatorname{erf} \varepsilon = 1 - \frac{1}{\sqrt{\pi}} \int_{\varepsilon^2}^{\infty} e^{-\tau}\tau^{-1/2}\, d\tau. \tag{1.2.10b}$$

We note that after integration by parts once (1.2.10b) becomes

$$\operatorname{erf} \varepsilon = 1 - \frac{1}{\sqrt{\pi}} \left[\frac{e^{-\varepsilon^2}}{\varepsilon} - \frac{1}{2} \int_{\varepsilon^2}^{\infty} e^{-\tau}\tau^{-3/2}\, d\tau \right]$$

and this suggests repeating the process in order to generate an expansion in increasing powers of ε^{-1}. If such an expansion were asymptotic in the limit $\varepsilon \to \infty$, it would be useful for numerical evaluation of erf ε for ε large.

Defining

$$F_n(\varepsilon) = \int_{\varepsilon^2}^{\infty} e^{-\tau}\tau^{-(2n+1)/2}\, d\tau, \qquad n = 0, 1, 2, \ldots \tag{1.2.11}$$

and integrating $F_n(\varepsilon)$ by parts results in the recursion relation

$$F_n(\varepsilon) = \frac{e^{-\varepsilon^2}}{\varepsilon^{2n+1}} - \frac{(2n+1)}{2} F_{n+1}(\varepsilon), \qquad n = 0, 1, 2, \ldots \tag{1.2.12}$$

and this can be used to calculate the following *exact* result for F_0

$$F_0(\varepsilon) = e^{-\varepsilon^2}\left[\frac{1}{\varepsilon} - \frac{1}{2\varepsilon^3} + \frac{1.3}{2^2\varepsilon^5} + \cdots + \frac{(-1)^{n-1}1.3.5\ldots(2n-3)}{2^{n-1}\varepsilon^{2n-1}}\right] + (-1)^n \frac{1.3.5\ldots(2n-1)}{2^n} F_n(\varepsilon), \qquad n = 1, 2, \ldots. \tag{1.2.13}$$

Thus, (1.2.13) exhibits a formal series in ascending powers of ε^{-1} and an exact expression for the remainder if the series is truncated after n terms. To show that the bracketed expression in (1.2.13) is the asymptotic expansion of F_0, we must verify that (1.2.5) is satisfied, i.e., that

$$F_0(\varepsilon) - e^{-\varepsilon^2}\sum_{n=1}^{M}(-1)^{n-1}\frac{1.3.5\ldots(2n-3)}{2^{n-1}\varepsilon^{2n-1}} = o(\varepsilon^{-(2M+1)}) \tag{1.2.14}$$

as $\varepsilon \to \infty$.

According to (1.2.13), the above reduces to showing that $S_M(\varepsilon)$ defined by

$$S_M(\varepsilon) = \varepsilon^{2M-1}\frac{(-1)^M 1.3.5\ldots(2M-1)}{2^M} F_M(\varepsilon) \tag{1.2.15}$$

tends to zero as $\varepsilon \to \infty$.

This is easily accomplished once we note that

$$F_M(\varepsilon) \le \frac{1}{\varepsilon^{2M+1}}\int_{\varepsilon^2}^{\infty} e^{-\tau}\,d\tau = \frac{e^{-\varepsilon^2}}{\varepsilon^{2M+1}}. \tag{1.2.16}$$

Therefore,

$$|S_M(\varepsilon)| \le \frac{1.3.5\ldots(2M-1)}{2^M\varepsilon^2} e^{-\varepsilon^2} \tag{1.2.17}$$

and hence $S_M = o(1)$ as $\varepsilon \to \infty$.

We note that the asymptotic expansion

$$\operatorname{erf}\varepsilon \sim 1 - \frac{e^{-\varepsilon^2}}{\sqrt{\pi}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}1.3.5\ldots(2n-3)}{2^{n-1}\varepsilon^{2n-1}} \tag{1.2.18}$$

is *divergent* because the numerical value of the coefficients of ε^{-2n+1} in the series (1.2.18) becomes large as n increases. Actually, (1.2.13) provides an exact expression for the error resulting from using M terms of the expansion (1.2.18) to represent erf ε. It is easily verified that for any fixed ε there is an optimal integer M_0 in the sense that the error is a decreasing function of the number n of terms retained, as long as $n < M_0$. But, if one insists on retaining

$n > M_0$ terms, the error will increase with n. Moreover, M_0 increases with ε and the error of the series with M_0 terms decreases as ε increases. The above features are typical of divergent asymptotic expansions.

The reader may verify that for $\varepsilon = 2$ the series on (1.2.18) gives the best accuracy if 5 terms are used and that the error in this case is only 6.43×10^{-5}, which is remarkable since $\varepsilon = 2$ is not a large number.

Functions defined by integral representations also occur naturally in the solution of linear problems by transform techniques. Various methods have been developed for calculating the asymptotic behavior of such results. A discussion of this topic is beyond the scope of this book. The reader will find an excellent account in Reference 1.2.1.

Problems

1. Calculate the asymptotic behavior as $t \to \infty$ for the initial value problem

$$\frac{d^2y}{dt^2} + y = \frac{1}{t}, \qquad \pi \leq t < \infty \tag{1.2.19}$$

$$y(\pi) = \frac{dy(\pi)}{dt} = 0 \tag{1.2.20}$$

by two methods.

(a) First, calculate the solution in integral form and use repeated integrations by parts.

(b) Next, observe that

$$y = a \sin t + b \cos t + \sum_{n=1}^{\infty} \frac{C_n}{t^n} \tag{1.2.21}$$

is formally, a general solution for appropriate C_n. Determine the constants a, b, and the C_n and compare your results with those in part (a). Is this asymptotic expansion convergent?

2. Noting that the nonlinear equation

$$\frac{d^2y}{dt^2} - \sin y = -\frac{1}{2} \tag{1.2.22}$$

has the energy integral

$$\frac{1}{2}\left(\frac{dy}{dt}\right)^2 + \cos y + \frac{y}{2} = E = \text{const.} \tag{1.2.23}$$

calculate the first five terms of the asymptotic expansion of the solution of (1.2.22) as $t \to \infty$ for the initial value problem

$$y(0) = 0$$

$$\frac{dy(0)}{dt} = 0$$

Reference

1.2.1 G. F. Carrier. M. Krook and C. E. Pearson, *Functions of a Complex Variable, Theory and Technique*, McGraw-Hill Book Company, New York, 1966.

1.3 Limit Process Expansions, Matching, General Asymptotic Expansions

Another possible way of defining a function $f(x, \varepsilon)$ is as the solution of a differential equation in which x is the independent variable and ε occurs as a parameter. If one cannot solve this differential equation for arbitrary ε (as, for example, if the differential equation is nonlinear with $\varepsilon \neq 0$) can one calculate the asymptotic expansion of the solution by considering a sequence of simpler differential equations governing each term of this expansion? This is the perturbation idea which will be explored in depth in subsequent chapters. Here, we consider a simple example to introduce some ideas.

The first-order equation

$$\varepsilon \frac{dy}{dx} + y = \frac{\varepsilon[x(\varepsilon - 1) + \varepsilon^2]e^{-x}}{(x + \varepsilon)^2}, \qquad 0 \leq x \leq \infty, 0 < \varepsilon \ll 1 \tag{1.3.1}$$

$$y(0) = 0 \tag{1.3.2}$$

has the exact solution

$$y = f(x, \varepsilon) \equiv e^{-x/\varepsilon} - \frac{\varepsilon e^{-x}}{x + \varepsilon}. \tag{1.3.3}$$

Ignoring temporarily the origin of eq. (1.3.3), we see that $f(x, \varepsilon)$ defines a well behaved function, and it is interesting to consider the asymptotic expansion of this function as $\varepsilon \to 0$. If we fix x to be some positive value and apply the limit process defined by eqs. (1.2.8) with $\phi_n(\varepsilon) = \varepsilon^n$ we find the following expansion for f, called an "outer" expansion

$$\begin{aligned} f &= -\varepsilon \frac{e^{-x}}{x} + \varepsilon^2 \frac{e^{-x}}{x^2} - \varepsilon^3 \frac{e^{-x}}{x^3} + O(\varepsilon^4) \\ &\equiv \sum_{n=0}^{N} \varepsilon^n h_n(x) + O(\varepsilon^{n+1}) \end{aligned} \tag{1.3.4}$$

and the contribution of the $e^{-x/\varepsilon}$ term is smaller than any term in the series in (1.3.4). We shall refer to such a term as a "transcendentally small" term (abbreviated as T.S.T.) in this limit.

Clearly (1.3.4) is not uniformly valid near $x = 0$. In fact, it is singular there, and this expansion is not a good approximation of the function defined by (1.3.3) no matter how small ε is if we allow x also to become small.

It is therefore natural to seek another expansion of (1.3.3) which adequately approximates this function near $x = 0$. Since the combination x/ε occurs in the first term one is led to the change of variables $x^* = x/\varepsilon$

$$y = g(x^*, \varepsilon) \equiv e^{-x^*} - \frac{e^{-\varepsilon x^*}}{x^* + 1}. \tag{1.3.5}$$

With $x^* = x/\varepsilon$ (1.3.5) defines the same function as f. However, the asymptotic expansion of g with x^* *fixed* as $\varepsilon \to 0$ is quite different. It is easy to see that in this limit y has the expansion which will be referred to as the "inner" expansion

$$\begin{aligned} g &= e^{-x^*} - \frac{1}{x^* + 1} + \frac{\varepsilon x^*}{x^* + 1} - \frac{\varepsilon^2 x^{*2}}{2(x^* + 1)} + \frac{\varepsilon^3 x^{*3}}{6(x^* + 1)} + O(\varepsilon^4) \\ &\equiv \sum_{n=0}^{N} \varepsilon^n g_n(x^*) + O(x^{*N+1}). \end{aligned} \tag{1.3.6}$$

Now, this expansion is accurate for small x. In particular, the condition $y = 0$ at $x = x^* = 0$ is satisfied. However, the result fails to be uniformly valid for x^* large. Thus, the two expansions (1.3.4) and (1.3.6) have mutually exclusive domains of validity. Hence, depending on the magnitude of x compared to ε one expansion or the other should be used.

Several related questions now arise.

(i) Equation (1.3.4) and (1.3.6) give the expansions of the *same* function by different limit processes. Is there another limit process expansion which is contained in both expansions?

(ii) Is it possible to find *one* asymptotic expansion which is uniformly valid for all $x \geq 0$?

(iii) Can one calculate these expansions directly from Equation (1.3.1) without knowing the exact solution?

We will now show that the answer to all these questions is in the affirmative for the present example. Building on the experience gained from this example we will introduce later on in this section the appropriate mathematical framework to define the above ideas.

Let us consider the expansion which would result from (1.3.1) by letting $\varepsilon \to 0$ with $x_\eta = x/\eta(\varepsilon)$ fixed for some $\eta(\varepsilon)$ such that $\varepsilon \ll \eta \ll 1$. Thus, in such an expansion $x \to 0$ in the limit but at a slower rate than in the case leading to (1.3.6). In a sense, the above defines an "intermediate" limit process.

Setting $x = \eta(\varepsilon)x_\eta$, we write (1.3.3) in the form

$$y = l(x_\eta, \eta, \varepsilon) \equiv e^{-\eta x_\eta/\varepsilon} - \frac{\varepsilon e^{-\eta x_\eta}}{\eta x_\eta + \varepsilon}. \tag{1.3.7}$$

Now, applying the limit process $\varepsilon \to 0$ with x_η fixed (1.3.7) has the following expansion, called the "intermediate" expansion

$$l = -\frac{\varepsilon}{\eta x_\eta} + \frac{\varepsilon^2}{\eta^2 x_\eta^2} - \frac{\varepsilon^3}{\eta^3 x_\eta^3} + \varepsilon - \frac{\varepsilon^2}{\eta x_\eta} - \frac{\varepsilon \eta x_\eta}{2} + O\left(\frac{\varepsilon^4}{\eta^4}\right) + O\left(\frac{\varepsilon^3}{\eta^2}\right) + O(\varepsilon^2) + O(\varepsilon\eta^2). \tag{1.3.8}$$

In the above the term $e^{-\eta x_\eta/\varepsilon}$ is transcendentally small and does not appear as long as $\varepsilon|\log \varepsilon| \ll \eta$. Henceforth, we shall always ignore such a term and automatically require that $\varepsilon|\log \varepsilon| \ll \eta$ in our calculations. This restricts somewhat the range of η to $\varepsilon|\log \varepsilon| \ll \eta \ll 1$.

If we now reexpand the outer and inner expansions using the above intermediate limit process, we find that if a sufficient number of terms are included both eventually give (1.3.8). This means that the outer expansion, which was constructed under the assumption $\varepsilon \to 0$ with x fixed $\neq 0$ is actually valid in the extended sense $\varepsilon \to 0$, $x_\eta = x/\eta(\varepsilon)$ fixed for some class of functions $\eta(\varepsilon) \ll 1$. Similarly, the inner expansion which was constructed under the assumption $\varepsilon \to 0$, $x^* = x/\varepsilon$ fixed $\neq \infty$ is actually valid in the extended sense $\varepsilon \to 0$, $x_\eta = x/\eta(\varepsilon)$ fixed, for some class of functions $\eta(\varepsilon)$ such that $\varepsilon|\log \varepsilon| \ll \eta$.

We will demonstrate next that for this example, the extended domains of validity of the inner and outer expansions overlap in the following sense. For each $R = 0, 1, 2, \ldots$ there exist integers P, and Q, and functions $\eta_1(\varepsilon)$ and $\eta_2(\varepsilon)$ with $\eta_1 \ll \eta_2$ such that

$$\lim_{\substack{\varepsilon \to 0 \\ x_\eta \text{ fixed}}} \frac{\left[\sum_{n=0}^{P} h_n(\eta x_\eta)\varepsilon^n - \sum_{n=0}^{Q} g_n(\eta x_\eta/\varepsilon)\varepsilon^n\right]}{\varepsilon^R} = 0 \tag{1.3.9}$$

for all η satisfying $\eta_1 \ll \eta \ll \eta_2$.

Equation (1.3.9) is a matching condition for the inner and outer expansions in their common overlap domain of validity which is defined as the class of functions $\eta(\varepsilon)$ satisfying the condition $\eta_1 \ll \eta \ll \eta_2$.

To demonstrate the result let us first take $R = 0$. Now $h_0 = 0$ and $g_0 = e^{-x^*} - 1/(1 + x^*)$. Assuming that $P = Q = 0$, the question now is whether

$$\lim_{\substack{\varepsilon \to 0 \\ x_\eta \text{ fixed}}} [h_0(\eta x_\eta) - g_0(\eta x_\eta/\varepsilon)] = 0. \tag{1.3.10}$$

Expanding g_0 in terms of x_η gives[1]

$$g_0 = -\frac{\varepsilon}{\eta x_\eta} + \frac{\varepsilon^2}{\eta^2 x_\eta^2} - \frac{\varepsilon^3}{\eta^3 x_\eta^3} + O\left(\frac{\varepsilon^4}{\eta^4}\right) + \text{T.S.T.} \tag{1.3.11}$$

[1] Note that the first two terms in this expansion correspond to the first two terms in the intermediate expansion (1.3.8).