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PROBLEMS IN PLANE GEOMETRY WITH SOLUTIONS

HORBLIT AND NIELSEN

The Student's Private Tutor

TRADE MARK REG.

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PROBLEMS IN PLANE GEOMETRY

WITH SOLUTIONS

By

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PREFACE

This is a problem book in Plane Geometry; it contains more than 300 problems and their solutions. The purpose of this book is manifold. Originally designed to aid students in their preparation for college board, Annapolis, and West Point entrance examinations, the manuscript was expanded and revised so that it may be used profitably by anyone who finds a need to review, test, or expand his knowledge of this fundamental subject of mathematics. It is well known that a true test of your knowledge of any mathematical subject is the manner in which you handle original problems, but equally important is the additional knowledge which comes with the solution of any problem. It may be said that the solution of problems gives added experience and in any endeavor there is no substitute for experience.

If you are contemplating taking any kind of examination which requires knowledge of Plane Geometry, this book will prove invaluable in your preparation; but, as has been said, it has other purposes. It may be used by students and teachers alike as a reference book for source material to supplement the work in any class in Plane Geometry. Draftsmen and designers will find it useful as a "brush-up" on their two-dimensional geometric concepts. College students will find that it serves as a good review of their high school mathematics.

The book is divided into two parts. The initial chapters of Part I deal with the technique of solving geometrical problems. Special comments are made on the solutions of construction and loci problems and examples are given throughout the discussion. The subject matter has been divided into five parts: circles, constructions, loci, polygons, and triangles. A chapter has been given to each of these parts. The beginning of each of these chapters contains some illustrative examples before the problems are listed. The problems are arranged more or less in order of their difficulty and therefore it is advantageous to take the problems in order. Part II contains the solutions of the problems stated in Part I. The solutions list each step of the problem and sufficient reasons are given so that the reader should have no difficulty in following the logic of the steps. A figure accompanies each problem so that the solution can be followed visually.

The authors gratefully acknowledge their indebtedness to Mrs. Carlene Nielsen for her aid in the preparation of the manuscript, to Mrs. Gladys Walterhouse for her careful reading and checking of the manuscript, and to the staff of Barnes and Noble, Inc., for their pleasant co-operation in the publication of this book with the minimum of typographical errors.

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Part One

TECHNIQUE OF PROBLEM SOLVING
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ABBREVIATIONS AND SYMBOLS

A′, *A* prime.
Adj., adjacent.
Alt., alternate; altitude.
Bis., bisector.
Comp., complementary.
Cons., construction.
Cor., corollary.
Corr., corresponding.
Def., definition.
Ex., exercise.
Ext., exterior.
Fig., figure.
Ft., feet.
Hyp., hypotenuse.
In., inch(es).
Int., interior.
Opp., opposite.
P_i, *P* sub *i*.
Prop., proposition.
Quad., quadrilateral.
Rect., rectangle.
Rt., right.
St., straight.
Supp., supplementary.

$=$, is equal to; equals.
$\neq$, is not equal to.
$\sim$, is similar to.
$\cong$, is congruent to; congruent.
$>$, is greater than.
$<$, is less than.
$\parallel$, is parallel to; parallel.
$\therefore$, therefore.
$\perp$, is perpendicular to; perpendicular.
. . ., and so on.
$\angle$, angle.
∡, angles.
⌒, arc.
$\odot$, circle.
Ⓢ, circles.
▱, parallelogram.
▱ with s, parallelograms.
▭, rectangle.
▭ with s, rectangles.
$\sqrt{n}$, square root of *n*.
$\triangle$, triangle.
△ with s, triangles.
′, feet; minutes.
″, inch(es); seconds.

s.a.s. = *s.a.s.* Two triangles are congruent if two sides and the included angles of one are equal, respectively, to two sides and the included angle of the other.

a.s.a. = *a.s.a.* Two triangles are congruent if two angles and the included side of one are equal, respectively, to two angles and the included side of the other.

s.s.s. = *s.s.s.* Two triangles are congruent if the sides of one are equal, respectively, to the sides of the other.

CHAPTER I

TECHNIQUE OF SOLVING ORIGINAL PROBLEMS

Introduction.

There is no cut-and-dried method of doing original problems in Geometry. As its very name implies, an original problem calls for ingenuity, mental agility, and insight. However, it is not true that those who lack these qualities are precluded from successfully solving geometrical problems; for there are certain general rules and directions which, if carefully followed, will be found very helpful in the solutions of problems. These general rules may be divided into four classes, to which we shall give the following names: (a) *the forward movement;* (b) *forcing the issue;* (c) *reductio ad absurdum;* (d) *the backward movement.* Before proceeding to a detailed discussion of the rules it should be pointed out that a careful study of the examples will also greatly benefit the student and help to develop a technique. However, may he be cautioned not to read the examples carelessly but to study them in detail.

The Forward Movement.

This method of attack may be called the natural method. You start with the given and then move forward, drawing deductions from the given and deductions from your deductions, until you arrive at the final conclusion, the Q.E.D. The method will be illustrated by some examples.

Example 1. If a median of a triangle is drawn, prove that the perpendicular from the other vertices upon this median are equal.

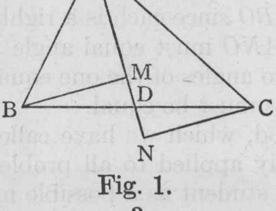

Fig. 1.

Solution. In Fig. 1, it is given that AD is a median to BC, and that BM and CN are perpendiculars from B and C respectively to AD or AD extended.

From the definition of the word "median," it is deduced that $BD = DC$; furthermore, since it is given that BM and CN are perpendiculars, we deduce that triangles BMD and CND are right triangles in which BD and DC are hypotenuses. Since BC and AN intersect at D, $\angle BDM = \angle NDC$; therefore, we conclude that, since the right triangles have the hypotenuse and an acute angle of one equal to the hypotenuse and an acute angle of the other, the triangles are congruent. Hence, we conclude that BM is equal to CN, since corresponding parts of congruent triangles are equal. This proves the statement of the problem.

Sometimes, the forward method involves the drawing of construction lines.

Example 2. If NO is the base of the isosceles triangle MNO and if the perpendicular from N to MO meets MO at A, prove that the angle ANO is equal to one-half the angle at M.

Solution. It is given that MNO is an isosceles triangle and that NA is perpendicular to MO. See Fig. 2.

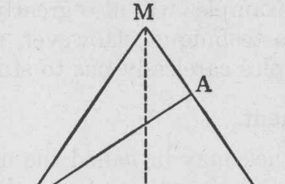

Fig. 2.

Since it is asked to prove that $\angle ANO = \frac{1}{2} \angle NMO$, it is natural to draw MB bisecting angle NMO. The problem now is to prove that $\angle ANO = \angle BMO$. Since MB is the bisector of the vertex angle of an isosceles triangle, it is perpendicular to the base, NO. In the two triangles MBO and NAO, angle AON is common, and angle NAO equals angle MBO since each is a right angle. Therefore we conclude that angle ANO must equal angle BMO, because, when two triangles have two angles of the one equal to two angles of the other, the third angles must be equal.

This natural method, which we have called the forward movement, cannot be easily applied to all problems, but it should be borne in mind by the student as a possible method of attack.

Forcing the Issue.

The second method, which we have called "forcing the issue," is really a variation of the first method. It is generally used when relationships between angles or lines are in question. To make this method clear let us again consider Example 2 above.

Solution. Since we are to prove $\angle ANO = \frac{1}{2} \angle NMO$, we ask ourselves what angle ANO does equal. From observation of the right triangle NAO, it is obvious that

$$\angle ANO = 90° - \angle NOA.$$

Our aim now should be to change the right side of this equation by the method of substituting equals for equals until it is ultimately changed into one-half of the angle NMO. Since the right side must contain only the angle NMO, the substitutions should be made with that in mind. First, $\angle NAO = 90°$, therefore the equation becomes

$$\angle ANO = \angle NAO - \angle NOA.$$

Since $\angle NAO$ is the exterior angle of triangle NMA, it therefore equals $\angle NMA + \angle MNA$. Thus

$$\angle ANO = \angle NMA + \angle MNA - \angle NOA.$$

Notice further that $\angle MNA = \angle MNO - \angle ANO$ and by substitution the original equation becomes

$$\angle ANO = \angle NMA + \angle MNO - \angle ANO - \angle NOA$$

and since $\angle MNO = \angle NOA$,

$$\angle ANO = \angle NMA - \angle ANO.$$

Solving this equation we get

$$2 \angle ANO = \angle NMA$$

or

$$\angle ANO = \tfrac{1}{2} \angle NMA. \quad \text{Q.E.D.}$$

In the above example we have thus "forced" the right side of the equation into the required form. Problem 40, Chapter VIII, is another example which lends itself to this type of solution. This method should usually be employed when it is a question about the relationship between one angle and another.

Reductio ad Absurdum.

The third method is called *reductio ad absurdum*, or, in plain English, it is a method of proving that, if the proposition in question is not true, an absurdity results.

Example 3. Consider the proposition that two straight lines can intersect only once.

Solution. Let the given straight lines, AB and CD, intersect at P.

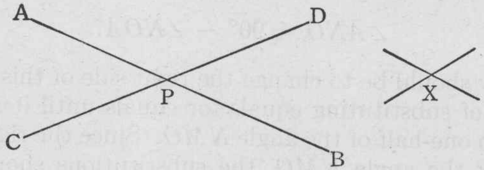

Fig. 3.

If we assume that the proposition is not true, in other words, that the two straight lines will intersect again, say at X, the result will be that between the two points, P and X, we shall have two straight lines, PDX and PBX. However, this is an absurdity, since between two points there can be only one straight line. Therefore, since the assumption that the proposition is not true leads to an absurdity, the proposition must be true.

Problem 45, Chapter VIII, is a good example of a problem to which this method is applicable. The method is of limited application, but it should be borne in mind.

The Backward Movement.

The last and most important method has been called the *backward movement* because we assume the conclusion and then draw deductions from the conclusion until we arrive at something known or something which can be easily proved. In other words this method is the opposite of the forward movement in which we begin with the given and proceed forward to the conclusion; in the backward movement we begin with the conclusion and move backward till we arrive at the given or the known. After we arrive at the given or the known, we then reverse the movement and proceed forward to the conclusion. Some examples will make this clear.

Example 4. Given that $ABCD$ is a rhombus and BD is a diagonal. Prove that $\angle m = \angle n$. See Fig. 4.

Solution. First assume that $\angle m = \angle n$. Now $AB \parallel DC$, since the figure is a rhombus, and consequently, $\angle m = \angle p$. Therefore, $\angle p = \angle n$ since we assumed that $\angle m = \angle n$; but if $\angle p = \angle n$,

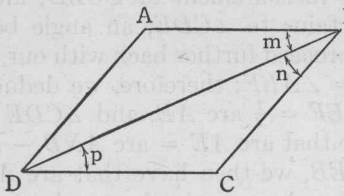

Fig. 4.

then DC must equal BC. We have now arrived at a known or a given, for we know that $DC = BC$ since the figure is a rhombus. The backward movement is now ended and we are ready to give the final proof by reversing our steps as follows:

$DC = BC$ since the figure is a rhombus; therefore $\triangle DCB$ is isosceles and $\angle p = \angle n$; but $\angle p = \angle m$ since $AB \parallel DC$; therefore $\angle m = \angle n$.

Example 5. Given: AOB is a diameter of the circle O; BM is tangent to the circle at B; CF is tangent to the circle at E and meets BM at C; the chord AE, when extended, meets BM at D. Prove that $BC = CD$. See Fig. 5.

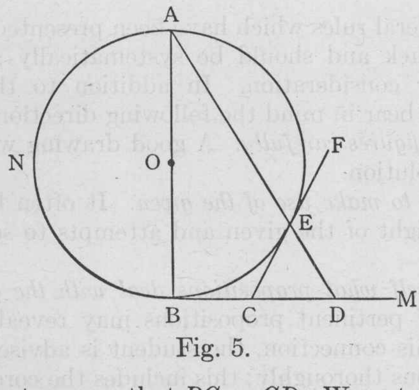

Fig. 5.

Solution. First assume that $BC = CD$. We note that $CE = BC$ since tangents from one point to a circle are equal. The first deduction, therefore, is that, since by assumption $BC = CD$, CE must equal CD. This deduction, however, does not lead to a known or a given, since we do not yet know that $CE = CD$.

Therefore, we proceed further with our deductions. The next deduction is that, since $CE = CD$, $\angle CED = \angle CDE$. This deduction cannot be directly confirmed since there is no theorem which gives us the measurement of $\angle CED$, although there is a theorem which pertains to $\angle CDE$, an angle between a tangent and a secant. We proceed further back with our deductions. Next note that $\angle CED = \angle AEF$; therefore, we deduce that $\angle AEF = \angle CDE$. Since $\angle AEF = \frac{1}{2}$ arc AE, and $\angle CDE = \frac{1}{2}$(arc ANB − arc BE), we deduce that arc AE = arc ANB − arc BE; but since arc ANB = arc AEB, we then have that arc AE = arc AEB − arc BE. At last we have arrived at a known, for arc AE + arc BE = arc AEB, or arc AE = arc AEB − arc BE. We have, therefore, reached the end of the backward movement and we are ready to reverse the steps for the regular proof. These steps are as follows:

Arc AE + arc BE = arc AEB; arc AE = arc AEB − arc BE; arc AE = arc ANB − arc BE; $\frac{1}{2}$ arc AE = $\frac{1}{2}$(arc ANB − arc BE); $\angle AEF = \angle CDE$; $\angle AEF = \angle CED$; therefore $\angle CED = \angle CDE$; therefore, $CE = CD$; but $CE = BC$; therefore, $BC = CD$.

This method is sometimes called the *analytic method.* It is very important and deserves the student's closest study.

Other Rules.

The four general rules which have been presented pertain to the method of attack and should be systematically applied to any problem under consideration. In addition to these rules, the student should bear in mind the following directions:

1. *Draw the figures carefully.* A good drawing will often reveal a clue to the solution.

2. *Remember to make use of the given.* It often happens that a student loses sight of the given and attempts to solve a problem without it.

3. *Ask yourself what propositions deal with the given.* A mere enumeration of pertinent propositions may reveal a clue to the solution. In this connection, the student is advised to know the book propositions thoroughly; this includes the corollaries.

4. *Note the significance of certain facts.* For example:

If one line is twice as long as another, there may be a 90°, 60°, 30° triangle present; or the shorter line may connect the midpoints of two sides of a triangle.

If the $\sqrt{3}$ appears, there may be a 90°, 60°, 30° triangle present.

If the $\sqrt{2}$ appears, there may be a 90°, 45°, 45° triangle present.

If the three sides of a triangle are given, it may be of advantage to find out whether the square of the longest side is equal to the sum of the squares of the other two sides.

If an equality of two products is given ($\overline{AB} \times \overline{CD} = \overline{EF} \times \overline{GH}$), rewrite the equation in the form of a proportion ($AB : EF = GH : CD$), in order to discover possible similar triangles or polygons.

The systematic application of the four general methods of attack and the observance of the more specific directions given above should be very helpful to the student in his geometric endeavors.

Presentation of a Solution.

Thus far in the examples no attempt has been made to present the solutions in a systematic form. We have been more interested in the discussion of a method of attack than in the actual solutions of the examples. However, it cannot be too strongly emphasized that in the presentation of a solution, a systematic form should be followed. Such a form will develop clear, logical thinking as well as present an orderly, easy-to-read solution. Most textbooks on Plane Geometry present the proof of the propositions in a standard form listing the steps (conclusions or deductions) in one column and the reasons in a second column. It may not always be possible to follow this form, or any standard form, in the solutions of problems, but an attempt should be made to have a well-organized solution.

A good form should have the following characteristics:

1. Statement of the given or hypothesis.
2. Statement of the conclusion or question specified in the problem.
3. Organization of the steps which arrive at the Q.E.D.
4. Organization of the reasons for the steps.
5. Specific indication of any constructions.

Two examples will be given here to illustrate the possible form. Other examples throughout the book may also be studied for their form.