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THE FOUNDATIONS OF DIFFERENTIAL GEOMETRY

BY
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AND
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PREFACE

THIS is intended as a companion to the Cambridge Tract No. 24, on Invariants of Quadratic Differential Forms. As its name implies it contains a set of axioms for differential geometry and develops their consequences up to a point where a more advanced book might reasonably begin. Formulae appear only incidentally and the reader is supposed to obtain those needed from the tract No. 24, or from other books and articles on the formal side of the subject.

Analytical operations with coordinate systems are continually used in differential geometry, a typical process being to "choose a coordinate system such that...." It is therefore natural to state the axioms in terms of an undefined class of "allowable" coordinate systems, and to deduce the properties of the space from the nature of the transformations of coordinates permitted by the axioms.

The axioms for differential geometry in general are preceded by more special sets of axioms in which the structure of a space is defined by an appropriate class of "preferred" coordinate systems. Thus Euclidean geometry is characterized by the class of rectangular cartesian coordinate systems. The "preferred" coordinate systems constitute a sub-class of the "allowable" coordinate systems for any one of these spaces. The former class is small, so as to characterize the structure of the space, and the latter is large, so as to permit freedom of analytic operation.

These earlier axioms are found to be adequate for the differential geometry of an open simply connected space, the most elementary theorems of which occupy the greater part of Chaps. III–V. The more general axioms, in terms of allowable coordinate systems and without restrictions on the connectivity of the space, are given in Chap. VI. We believe that they provide an adequate foundation for any of the differential geometries which are now being studied. The complete theory which should be constructed out of these axioms would be a combination of infinitesimal geometry and analysis situs. In the final chapter we outline some of the questions which arise, in the hope that some of the readers of this tract may participate in the construction of a branch of mathematics which we are convinced is of great importance.

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J. H. C. W.

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CONTENTS

CHAPTER I

THE ARITHMETIC SPACE OF n DIMENSIONS

1. Arithmetic points.

Analysis habitually borrows not only terminology but also methods and results from geometry. In the present chapter we mean to indicate how this can be done without running into a vicious circle in the application of analysis to geometry. We shall be particularly concerned with the ideas clustering about the notion of linear dependence.

We shall presuppose the contents* of *Q.F.* Chap. I, and in particular Cramer's rule (*Q.F.* p. 6), by which a set of linear equations

$$(1\cdot1) \qquad y^i = \sum_j a^i_j x^j$$

can be solved to yield

$$(1\cdot2) \qquad x^i = \sum \alpha^i_j y^j,$$

provided the determinant

$$a = | a^i_j |$$

is not zero. With the notation of *Q.F.*,

$$(1\cdot3) \qquad \alpha^i_j = \frac{1}{a} A^i_j,$$

where A^i_j is the co-factor of the element a^j_i in the matrix $||a^i_j||$, and α^i_j is called the *normalized co-factor* of a^j_i.

An ordered set of n real† numbers $(x^1, \ldots, x^n)$ will be called an *arithmetic point*, and $x^1, \ldots, x^n$ its *components*. The set of all arithmetic points, for a given value of n, will be called the *arithmetic space of n dimensions*. As in *Q.F.* we shall denote an arithmetic point by a single letter x, and in considering several arithmetic points shall distinguish them by subscripts; thus x_a will stand for $(x^1_a, \ldots, x^n_a)$.

In Euclidean geometry, for instance, all points are alike, but in the arithmetic space each point has an individuality of its own. In particular the points $(0, \ldots, 0)$, and $(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$ will be called

* This book is intended to run parallel to the *Cambridge Tract*, No. 24, by O. Veblen, called *Invariants of Quadratic Differential Forms*, which will be referred to as *Q.F.*

† There is nothing to prevent our taking numbers from any field, but we shall be content to use the real number system of analysis.

the origin and unit points respectively. We shall denote the origin by e_0 and the unit points by $e_1, \ldots, e_n$.

In many books on analysis an ordered set of numbers $(x^1, \ldots, x^n)$ is called simply a point (without any adjective). On the other hand in books on algebra* it is often called a vector. Two fundamental operations in vector algebra are multiplication by a number and addition. More precisely, if x is an arithmetic point (vector), and if a is a number, then ax is the point

$$ax = (ax^1, \ldots, ax^n),$$

while if x and y are two points, $x + y$ is the point

$$(x^1 + y^1, \ldots, x^n + y^n).$$

Combining these two operations we can define the difference of two points, and in general any linear combination,

$$t^1 x_1 + \ldots + t^k x_k,$$

of k points, $x_1, \ldots, x_k$. The theory of linear dependence has to do with properties which can be stated in terms of these two operations.

2. Linear dependence.

The points given by†

$$(2{\cdot}1) \qquad x^i = t^\alpha x^i_\alpha, \qquad (\alpha = 1, \ldots, m)$$

are said to be *linearly dependent* on $x_1, \ldots, x_m$. According to this definition the origin is linearly dependent on any set of points. A set of two or more points is said to be *linearly independent* if no one of them is dependent on the rest. To complete the definition we say that a single point is linearly independent if it is not the origin.

A set of points, $x_1, \ldots, x_m$, is independent if, and only if, the relation

$$(2{\cdot}2) \qquad s^\alpha x^i_\alpha = 0$$

implies $s^1 = \ldots = s^m = 0$. For if one of these coefficients s^1, say, did not vanish, (2·2) would give

$$x^j_1 = -\frac{s^2}{s^1} x^j_2 - \ldots - \frac{s^m}{s^1} x^j_m.$$

A relation of the form (2·1), on the other hand, is a special case of (2·2).

Let $a_1, \ldots, a_m$ be any set of points. Either $a_1, \ldots, a_m$ all coincide with

* E. Study, *Einleitung in die Theorie der Invarianten linearer Transformation auf Grund der Vektorenrechnung*, Braunschweig, 1923; also H. Weyl, *Gruppentheorie und Quantenmechanik*, Leipzig, 1931.

† As in *Q. F.* a repeated index will imply summation. Roman indices will invariably run from 1 to n, while the range of Greek indices will be indicated in the text.

the origin or one of them, a_1, say, is independent. Either they will all depend upon a_1 or the set a_1, a_2, say, will be independent. Proceeding in this way we shall arrive at an independent set $a_1, \ldots, a_p$, say, upon which all the points $a_1, \ldots, a_m$ will be linearly dependent. We need a criterion to determine p, and we get this by considering the matrix (*Q.F.* p. 4),

$$||a^i_\beta|| = \left\|\begin{matrix} a^1_1, & a^1_2, & \cdots & a^1_m \\ a^2_1, & a^2_2, & \cdots & a^2_m \\ \vdots & \vdots & & \vdots \\ a^n_1, & a^n_2, & \cdots & a^n_m \end{matrix}\right\|,$$

whose columns are the m arithmetic points.

If all the $(p+1)$-row determinants (*Q.F.* p. 9) vanish, but at least one p-row determinant does not, the matrix is said to be of *rank* p. The fundamental theorem of linear dependence is:

If p is the rank of the matrix $||a^i_\beta||$, the points $a_1, \ldots, a_m$ are all dependent upon p of them which are themselves independent.

To prove this, first consider the case where $m \leqq n$. The matrix $||a^i_\beta||$ is of rank p, and without loss of generality we may suppose the determinant

$$a = |a^\lambda_\mu|, \qquad (\lambda, \mu = 1, \ldots, p)$$

to differ from zero. If $p = m$ it follows from Cramer's rule for solving linear equations that the points $a_1, \ldots, a_m$ are independent. For, since $a \neq 0$, the equations

$$a^\alpha_\beta x^\beta = 0, \qquad (\alpha, \beta = 1, \ldots, m)$$

have the unique set of solutions $(0, \ldots, 0)$.

If $p < m$ the points $a_1, \ldots, a_p$ are shown to be independent by the argument we have just used. Let $A^1_\sigma, \ldots, A^p_\sigma$ be the co-factors of $a^i_1, \ldots, a^i_p$ in the matrix

$$\left\|\begin{matrix} a^1_1, & \cdots & a^1_p, & a^1_\sigma \\ \vdots & & \vdots & \vdots \\ a^p_1, & \cdots & a^p_p, & a^p_\sigma \\ a^i_1, & \cdots & a^i_p, & a^i_\sigma \end{matrix}\right\|.$$

The determinant of this matrix is

$$(-1)^p a a^i_\sigma + A^\lambda_\sigma a^i_\lambda, \qquad (\lambda = 1, \ldots, p).$$

For $i \leqq p$ this vanishes since two rows have equal elements, and for $i > p$ it vanishes since the rank of $||a^i_\beta||$ is p. The coefficients A^λ_σ and a do not depend on the elements $a^i_\sigma, a^i_1, \ldots, a^i_p$, and so, writing

$$A^\lambda_\sigma / a = (-1)^{p-1} x^\lambda_\sigma,$$

we have

$$(2\cdot3) \qquad a^i_\sigma = x^\lambda_\sigma a^i_\lambda, \qquad (\sigma = p+1, \ldots, m).$$

If $m > n$ we consider the points $a_\beta = (a^1_\beta, \ldots, a^n_\beta, 0, \ldots, 0)$, in the arithmetic space of m dimensions. We can then apply the above argument to obtain the relation (2·3), and the theorem is established for all values of m.

3. Linear sub-spaces.

If $x_1, \ldots, x_k$ are k linearly independent points, the set of points linearly dependent on them will be called an *arithmetic linear k-space**, and the points $x_1, \ldots, x_k$ will be said to *span* the linear k-space defined in this way. Thus a linear 1-space consists of the points whose components are proportional to those of a given point, and may conveniently be called an arithmetic straight line through the origin.

From the equations

$$(3\cdot1) \qquad x^i = t^\lambda x^i_\lambda, \qquad (\lambda = 1, \ldots, k)$$

which define a linear k-space, X_k, it follows that to each point $(t^1, \ldots, t^k)$ of the arithmetic space of k dimensions corresponds a point of X_k, the points e_0 and $e_1, \ldots, e_k$ corresponding to the origin and $x_1, \ldots, x_k$ respectively. Moreover, to each point of X_k corresponds just one point in the k-dimensional arithmetic space. For if t_1 and t_2 are points in the latter corresponding to the same point in X_k, we have

$$t^\lambda_1 x^i_\lambda = t^\lambda_2 x^i_\lambda,$$

or

$$(t^\lambda_2 - t^\lambda_1) x^i_\lambda = 0.$$

But $x_1, \ldots, x_k$ are independent and so $t_1 = t_2$.

Equations of the form (3·1), therefore, define not only a linear k-space, but a linear k-space which is in (1-1) correspondence with the arithmetic space of k dimensions. Such a correspondence is called a *parameterization* of the linear k-space.

All points linearly dependent on m points $a_1, \ldots, a_m$, in a linear k-space, X_k, are contained in X_k. For $a_1, \ldots, a_m$ are given by equations of the form

$$a^i_\beta = t^\lambda_\beta x^i_\lambda,$$

* We shall define flat sub-spaces in general in § 7 below. The linear sub-spaces all contain the origin. They owe their importance to the fact that (with the notations explained in § 1) if a linear k-space contains two points x_1 and x_2, it also contains $x_1 + x_2$. We can express this by saying that linear k-spaces are closed under addition.

and any point linearly dependent on $a_1, \ldots, a_m$ is obviously dependent on $x_1, \ldots, x_k$. This may be called the transitive law for linear dependence.

If the points $x_1, \ldots, x_k$ span the linear k-space, X_k, there is no other linear k-space containing these points. For if $y_1, \ldots, y_k$ span a linear k-space, Y_k, containing $x_1, \ldots, x_k$, we have

$$x^i_\lambda = t^\mu_\lambda y^i_\mu. \tag{3·2}$$

If the determinant $| t^\lambda_\mu |$ were zero there would be a relation of the form

$$s^\mu t^\lambda_\mu = 0,$$

in which $s^1, \ldots, s^k$ were not all zero. This would imply

$$s^\lambda x^i_\lambda = 0$$

and $x_1, \ldots, x_k$ would not be independent. For each value of i, therefore, the equations (3·2) can be solved by Cramer's rule to yield equations of the form

$$y^i_\lambda = T^\mu_\lambda x^i_\mu. \tag{3·3}$$

From the transitive law for linear dependence, and from (3·2) it follows that each point of X_k lies in Y_k. Similarly it follows from (3·3) that each point of Y_k lies in X_k. They are, therefore, identical. We can express this by saying that a linear k-space is spanned by any set of k independent points contained in it, and it follows that a linear k-space does not contain a set of l independent points, where $l > k$. For the definition in § 2 implies that any k points in a set of l independent points are themselves independent, and would therefore span any linear k-space containing the larger set.

The theorem of § 2 can now be stated in the form: *If p is the rank of a matrix $\| x^i_\beta \|$, the points $x_1, \ldots, x_m$ are all contained in a linear p-space but not in a linear q-space, where $q < p$.*

4. Linear homogeneous transformations.

Any correspondence under which each point, x, in a set of points X corresponds to a unique point, y, is called a *single valued transformation* of X into Y, where Y is the set of points to which the points of X correspond. We may denote the transformation by

$$x \to y.$$

If no two distinct points in X correspond to the same point in Y, the transformation $x \to y$ will be called (1-1), or *non-singular*. If $x \to y$ is any non-singular transformation there exists a unique single-valued transformation, $y \to x$, called the *inverse* of $x \to y$.

A transformation which is given by equations of the form

$$y^i = a^i_j x^j \tag{4·1}$$

is said to be *linear* and *homogeneous*, and is non-singular if the determinant $a = |a^i_j|$ is not zero. For if $a \neq 0$ the equations (4·1), which are identical with (1·1), can be solved to obtain the inverse transformation

$$x^i = \alpha^i_j y^j, \tag{4·2}$$

where α^i_j is the normalized co-factor of a^j_i.

Any linear transformation $x \to y$, whether singular or not, will carry any point which is dependent upon a given set, $x_1, \ldots, x_k$, into a point which is dependent upon $y_1, \ldots, y_k$, where $x_\lambda \to y_\lambda$. For a point x, given by

$$x^i = t^\lambda x^i_\lambda, \tag{4·3}$$

goes into a point y, given by

$$y^i = a^i_j x^j = a^i_j t^\lambda x^j_\lambda = t^\lambda y^i_\lambda. \tag{4·4}$$

Not only the relation of linear dependence, therefore, but also the parameters $t^1, \ldots, t^k$, by which it is expressed, are unaltered by linear transformations.

The unit points are carried into the columns of the matrix $||a^i_j||$, and if p is the rank of the latter these columns will be contained in a linear p-space X_p, exactly p of them being independent. Hence the whole arithmetic space will be carried into X_p, and any two points, x_1 and x_2, such that

$$a^i_j (x^j_2 - x^j_1) = 0,$$

will be carried into the same point in X_p. The condition $a \neq 0$ is, therefore, not only sufficient, but also necessary in order that the transformation given by (4·1) shall be non-singular.

An independent set of points, $x_1, \ldots, x_k$, is carried by a non-singular linear homogeneous transformation*, $x \to y$, into an independent set $y_1, \ldots, y_k$. For if some of the latter were dependent upon the others we could apply the inverse transformation, $y \to x$, to show that the same was true of $x_1, \ldots, x_k$. Linear homogeneous transformations, therefore, carry linear k-spaces into linear k-spaces, and from the theorem in § 2 it follows that the matrices

$$||x^i_\alpha||, \text{ and } ||y^i_\alpha|| = ||a^i_j x^j_\alpha||,$$

have the same rank.

* In this and the following chapters, all transformations are to be taken as non-singular unless the contrary is stated.

Cramer's rule depends upon the fact that if a^k_j are given, and if $a \neq 0$, there exists a matrix $||\alpha_j||$ which is uniquely determined by the condition

$$\delta^i_j = \alpha^i_k \, a^k_j . \tag{4·5}$$

If we regard the columns of $||a^k_j||$ as arithmetic points this states that there is just one linear homogeneous transformation which carries a given set of n independent points, $a_1, \ldots, a_n$, into the unit points $e_1, \ldots, e_n$.

As a corollary, we see that there is at least one linear homogeneous transformation which carries any set of k independent points, $a_1, \ldots, a_k$, into the unit points $e_1, \ldots, e_k$. For we can find $n-k$ points $a_{k+1}, \ldots, a_n$ such that $a_1, \ldots, a_n$ are linearly independent (if the determinant $|a^\lambda_\mu|$, (λ, $\mu = 1, \ldots, k$) is not zero we can take $a_{k+1} = e_{k+1}, \ldots, a_n = e_n$) and there is just one transformation in which $a_i \to e_i$. A transformation which carries an independent set of points $a_1, \ldots, a_k$ into $e_1, \ldots, e_k$, will carry the linear k-space spanned by the former into that given by

$$\begin{cases} y^\lambda = t^\lambda, & (\lambda = 1, \ldots, k) \\ y^\sigma = 0, & (\sigma = k+1, \ldots, n) \end{cases} \tag{4·6}$$

that is to say, into the set of all points satisfying the equations

$$y^{k+1} = 0, \ldots, y^n = 0.$$

5. Homogeneous linear equations.

There is a linear homogeneous transformation

$$y^i = a^i_j \, x^j, \tag{5·1}$$

which carries a given linear k-space, X_k, into the linear k-space given by (4·6). It follows that X_k consists of those, and only those points, which satisfy the set of $n-k$ linear homogeneous equations

$$a^\sigma_j \, x^j = 0 \qquad (\sigma = k+1, \ldots, n). \tag{5·2}$$

Again, if (5·2) is any set of $n-k$ linear homogeneous equations in n variables, x, such that the matrix

$$||a^\sigma_j||$$

is of rank $n-k$, a transformation (5·1) can be found which carries the set of points satisfying (5·2) into the linear k-space (4·6). Since linear k-spaces are carried into linear k-spaces by linear homogeneous transformations, it follows that the solutions of (5·2) constitute a linear k-space.

If

(5·3) $$b_j x^j = 0, \qquad (\alpha = 1, \ldots, m)$$

is any set of linear homogeneous equations, such that the rank of the matrix $\|b_j^\alpha\|$ is r, there are r of these equations, which we may suppose to be

(5·4) $$b_j^\sigma x^j = 0, \qquad (\sigma = 1, \ldots, r)$$

such that the matrix $\|b_j^\sigma\|$ is of rank r, and the remaining equations are linear combinations of (5·4). Hence the points satisfying (5·4) satisfy the full set of equations, and of course, any point satisfying (5·3) satisfies (5·4). By the last paragraph the points satisfying (5·4) constitute a linear $(n-r)$-space. *Therefore the solutions to a set of linear homogeneous equations constitute an $(n-r)$-space, where r is the rank of the matrix of the coefficients.*

Any set of points which span the $(n-r)$-space is called a *complete set* of solutions.

Taken with the description of the way in which a linear k-space is spanned by sets of k independent points, this summarizes the theory of linear homogeneous equations.

6. Translations.

A transformation given by equations of the form

(6·1) $$y^i = x^i + a^i$$

is called a *translation*. It is obvious that translations are non-singular and that the inverse of a translation is a translation; also that if $x \to y$ and $y \to z$ are translations, the resultant transformation $x \to z$ is a translation; also that there is just one translation, namely that given by

$$y^i = x^i + y_0^i - x_0^i,$$

which carries a given point x_0 into a given point y_0.

7. Flat sub-spaces.

Any set of points which corresponds under a translation to a linear k-space will be called an *arithmetic flat k-space*. For $k=0$, an arithmetic flat k-space is a single point; for $k=1$ it is called an arithmetic straight line, for $k=2$ a plane, and for $k=n-1$ a hyperplane. If one of two flat k-spaces can be carried by a translation into the other, they are said to be *parallel*. From the transitive property of translations it follows that flat k-spaces are carried by translations into flat k-spaces, and that two flat k-spaces which are parallel to a third, are parallel to each other. Through any point x_0 there is a flat k-space parallel to a given linear k-space, X_k. This flat k-space is obtained from X_k by the translation which carries the origin into x_0. Any flat k-space is parallel to itself by this definition, since the identical transformation, which leaves each point unaltered, is a special case of a translation.

If we apply the translation given by

$$(7{\cdot}1) \qquad y^i = x^i + y_0^i$$

to the linear k-space whose points satisfy $(n-k)$ linearly independent linear homogeneous equations

$$(7{\cdot}2) \qquad a_j^\sigma x^j = 0, \qquad (\sigma = k+1, \ldots, n)$$

we find that any flat k-space is the set of points satisfying a set of equations of the form

$$(7{\cdot}3) \qquad a_j^\sigma y^j = a_0^\sigma .$$

The constants a_0 are given by

$$a_0^\sigma = a_j^\sigma y_0^j,$$

and will be zero if, and only if, the point y_0 to which the origin is carried by the translation is in the linear k-space (7·2). Similarly, if y_0 and y_0' are any points in the flat k-space (7·3), the translation $y_0 \rightarrow y_0'$ carries this flat k-space into itself. We recall that a flat k-space, Y_k, was defined as the image of a linear k-space, X_k, under a translation $e_0 \rightarrow y_0$, and it follows that Y_k is equally well defined as the image of X_k under the translation $e_0 \rightarrow y_0'$, where y_0' is any point in Y_k. Therefore any pair of flat k-spaces which are parallel to each other and have a common point are identical. Hence there is one, and only one, flat k-space which passes through a given point and is parallel to a given flat k-space.

Conversely, if we have a set of $n-k$ linear equations of the form (7·3), such that the matrix of the coefficients on the left,

$$\| a_j^\sigma \|,$$

is of rank $(n-k)$, they are satisfied by a set of points which constitute a k-space. To prove this, we first observe that we can transpose k of the variables, y, to the right of (7·3), leaving on the left $n-k$ variables, y, the determinant of whose coefficients is not zero. Then substitute arbitrary values for the y's on the right, solve for the remaining ones by Cramer's rule, and we have a set of values $y_0^1, \ldots, y_0^n$ which satisfy (7·3). Now apply the translation $y \rightarrow x$ inverse to (7·1), and we find that the set of points satisfying (7·3) is carried into the set of points satisfying (7·2). In other words, the solutions of (7·3) satisfy the definition of a flat k-space.

By definition, a linear k-space is a set of points satisfying the equations (3·1). Applying the translation (7·1) we find that a flat k-space in general is a set of points satisfying equations of the form

$$(7\cdot 4) \qquad y^i = y^i_0 + t^\lambda (y^i_\lambda - y^i_0), \qquad (\lambda = 1, \ldots, k).$$

Hence the theorem which we have just proved asserts that the solutions of a set of equations of the form (7·3) are given by equations of the form (7·4), and conversely, any set of points, y, given by equations of the type (7·4), are the solutions of a set of equations of the form (7·3).

The flat k-space (7·4) contains the points $y_0, y_1, \cdots, y_k$. Substituting the right-hand side of (7·4) into the right-hand side of the formula for a homogeneous linear transformation, $y \to z$, it follows that (7·4) is carried by $y \to z$ into a flat k-space through the points $z_0, z_1, \cdots, z_k$, where $y_\alpha \to z_\alpha$.

8. Non-homogeneous linear equations.

Consider a general set of linear equations

$$(8\cdot 1) \qquad a^\alpha_j x^j = a^\alpha_0, \qquad (\alpha = 1, \ldots, m)$$

and let us refer to

$$\| a^\alpha_j \|$$

as the matrix of the coefficients, and to the matrix

$$\| a^\alpha_\sigma \|, \qquad (\sigma = 0, 1, \cdots, n)$$

with the column a^α_0 adjoined, as the *augmented* matrix. Let r be the rank of the matrix of the coefficients and s the rank of the augmented matrix. Naturally, $s \geqq r$. We may assume without loss of generality that the rank of the matrix

$$\| a^\lambda_j \|, \qquad (\lambda = 1, \ldots, r)$$

of the first r of the equations (8·1) is r, and therefore that the equations

$$(8\cdot 2) \qquad a^\lambda_j x^j = a^\lambda_0$$

are satisfied by all the points in some flat $(n-r)$-space, and only by those. If $s > r$ there is an equation

$$(8\cdot 3) \qquad a_j x^j = a_0$$

in the set (8·1), such that no relation of the form

$$(8\cdot 4) \qquad a_\sigma = p_\lambda a^\lambda_\sigma, \qquad (\lambda = 1, \ldots, r;\ \sigma = 0, 1, \cdots, n)$$