

**Textbooks in Mathematical Sciences**

# **Understanding Nonlinear Dynamics**

**非线性动力学入门**

**Daniel Kaplan  
and  
Leon Glass**

**Springer-Verlag**

**世界图书出版公司**



# Understanding Nonlinear Dynamics

Daniel Kaplan □ Leon Glass

Department of Physiology

McGill University □ Montréal, Québec, Canada

---

**Springer-Verlag**

世界图书出版公司

北京·广州·上海·西安

书 名 :Understanding Nonlinear Dynamics  
作 者: D. Kaplan, L. Glass  
中译名: 非线性动力学入门  
出版者: 世界图书出版公司北京公司  
印刷者: 北京中西印刷厂  
发 行: 世界图书出版公司北京公司(北京朝内大街 137 号 100010)  
开 本: 大 32 印张: 13.75  
版 次: 1997 年 9 月第 1 版 1997 年 9 月第 1 次印刷  
书 号: ISBN 7-5062-3308-8/O · 187  
版权登记: 图字 01-97-0454  
定 价: 54.00 元

世界图书出版公司北京公司已获得 Springer-Verlag 授权在中国境内  
独家重印、发行。

## Textbooks in Mathematical Sciences

### Series Editors:

Thomas F. Banchoff  
*Brown University*

John Ewing  
*Indiana University*

Gaston Gonnet  
*ETH Zentrum, Zürich*

Jerrold Marsden  
*University of California, Berkeley*

Stan Wagon  
*Macalester College*

Library of Congress Cataloging-in-Publication Data  
Kaplan, Daniel. 1959–

Understanding nonlinear dynamics / Daniel Kaplan and Leon Glass.  
p. cm. – (Textbooks in mathematical sciences)

Includes bibliographical references and index.

ISBN 0-387-94440-0

1. Dynamics. 2. Nonlinear theories. I. Glass, Leon, 1943–. II. Title. III. Series.

QA845.K36 1995 94-43113

515'.352–dc20

Printed on acid-free paper.

© 1995 Springer-Verlag New York, Inc.

All rights reserved. This work may not be translated or copied in whole or in part without the written permission of the publisher (Springer-Verlag New York, Inc., 175 Fifth Avenue, New York, NY 10010, USA), except for brief excerpts in connection with reviews or scholarly analysis. Use in connection with any form of information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed is forbidden.

The use of general descriptive names, trade names, trademarks, etc., in this publication, even if the former are not especially identified, is not to be taken as a sign that such names, as understood by the Trade Marks and Merchandise Marks Act, may accordingly be used freely by anyone.

This reprint has been authorized by Springer-Verlag (Berlin/Heidelberg/New York) for sale in the People's Republic of China only and not for export therefrom.

Reprinted in China by Beijing World Publishing Corporation, 1997.

ISBN 0-387-94440-0 Springer-Verlag New York Berlin Heidelberg

To Maya and Tamar. — DTK

To Kathy, Hannah, and Paul  
and in memory of Daniel. — LG

**165860**

# Preface

This book is about *dynamics*—the mathematics of how things change in time. The universe around us presents a kaleidoscope of quantities that vary with time, ranging from the extragalactic pulsation of quasars to the fluctuations in sunspot activity on our sun; from the changing outdoor temperature associated with the four seasons to the daily temperature fluctuations in our bodies; from the incidence of infectious diseases such as measles to the tumultuous trend of stock prices.

Since 1984, some of the vocabulary of dynamics—such as *chaos*, *fractals*, and *nonlinear*—has evolved from abstruse terminology to a part of common language. In addition to a large technical scientific literature, the subjects these terms cover are the focus of many popular articles, books, and even novels. These popularizations have presented “chaos theory” as a scientific revolution. While this may be journalistic hyperbole, there is little question that many of the important concepts involved in modern dynamics—global multistability, local stability, sensitive dependence on initial conditions, attractors—are highly relevant to many areas of study including biology, engineering, medicine, ecology, economics, and astronomy.

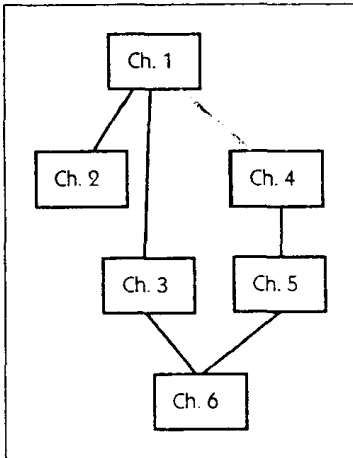
This book presents the main concepts and applications of nonlinear dynamics at an elementary level. The text is based on a one-semester undergraduate course that has been offered since 1975 at McGill University and that has been constantly updated to keep up with current developments. Most of the students enrolled in the course are studying biological sciences and have completed a year of calculus with no intention to study further mathematics. Since the main concepts of nonlinear dynamics are largely accessible using only elementary arguments, students are able to understand the mathematics and successfully carry out computations. The exciting nature and modernity of the concepts and the graphics are further stimuli that motivate students.

Mathematical developments since the mid 1970's have shown that many interesting phenomena can arise in simple finite-difference equations. These are introduced in Chapter 1, where the student is initiated into three important mathematical themes of the course: local stability analysis, global multistability, and problem solving using both an algebraic and a geometric approach. The graphical iteration of one-dimensional, finite-difference equations, combined with the analysis of the local stability of steady states, provides two complementary views of the same problem. The concept of chaos is introduced as soon as possible, after the student is able graphically to iterate a one-dimensional, finite-difference equation, and understands the concept of stability. For most students, this is the first exposure to mathematics from the twentieth century!

From the instructor's point of view, this topic offers the opportunity to refresh students' memory and skills in differential calculus. Since some students take this course several years after studying geometry and calculus, some skills have become rusty. Appendix A reviews important functions such as the Hill function, the Gaussian distribution, and the conic sections. Many exercises that can help in solidifying geometry and calculus skills are included in Appendix A.

Chapters 2 and 3 continue the study of discrete-time systems. Networks and cellular automata (Chapter 2) are important both from a conceptual and technical perspective, and because of their relevance to computers. The recent interest in neural and gene networks makes this an important area for applications and current research.

Many students are familiar with fractal images from the myriad popularizations of that topic. While the images provide a compelling motivation for studying nonlinear dynamics, the concepts of self-similarity and fractional dimension are important from a mathematical perspective. Chapter 3 discusses self-similarity and fractals in a way that is closely linked to the dynamics discussed in Chapter 1. Fractals arise from dynamics in many unexpected ways. The concept of a fractional dimension is unfamiliar initially but can be appreciated by those without advanced technical abilities. Recognizing the importance of computers in studying fractals, we use a computer-based notation in presenting some of the material.



Dependencies among the chapters.

The study of continuous-time systems forms much of the second half of the book. Chapter 4 deals with one-dimensional differential equations. Because of the importance of exponential growth and decay in applications, we believe that every science student should be exposed to the linear one-dimensional differential equation, learning what it means and how to solve it. In addition, it is essential that those interested in science appreciate the limitations that nonlinearities impose on exponential (“Malthusian”) growth. In Chapter 4, algebraic analysis of the linear stability of steady states of nonlinear equations is combined with the graphical analysis of the asymptotic dynamics of nonlinear equations to provide another exposure to the complementary use of algebraic and geometric methods of analysis.

Chapter 5 deals with differential equations with two variables. Such equations often appear in the context of compartmental models, which have been proposed in diverse fields including ion channel kinetics, pharmacokinetics, and ecological systems. The analysis of the stability of steady states in two-dimensional nonlinear equations and the geometric sketching of the trajectories in the phase plane provide the most challenging aspect of the course. However, the same basic conceptual approach is used here as is used in the linear stability analyses in Chapter 1 and Chapter 4, and the material can be presented using elementary methods only.

In most students’ mathematical education, a chasm exists between the concepts they learn and the applications in which they are interested. To help bridge this gap, Chapter 6 discusses methods of data analysis including classical methods (mean, standard deviation, the autocorrelation function) and modern methods derived from nonlinear dynamics (time-lag embeddings, dimension and related



topics). This chapter may be of particular interest to researchers interested in applying some of the concepts from nonlinear dynamics to their work.

In order to illustrate the practical use of concepts from dynamics in applications, we have punctuated the text with short essays called “Dynamics in Action.” These cover a wide diversity of subjects, ranging from the random drift of molecules to the deterministic patterns underlying global climate changes.

Following each chapter is supplementary material. The notes and references provide a guide to additional references that may be fun to read and are accessible to beginning students. A set of exercises reviewing concepts and mathematical skills is also provided for each chapter. Solutions to selected exercises are provided at the end of the book. For each chapter, we also give a set of computer exercises. The computer exercises introduce students to some of the ways computers can be used in nonlinear dynamics. The computer exercises can provide many opportunities for a term project for students.

The appropriate use of this book in a course depends on the student clientele and the orientation of the instructors. In our instruction of biological science students at McGill, emphasis has been on developing analytical and geometrical skills to carry out stability analysis and analysis of asymptotic dynamics in one-dimensional finite-difference equations and in one- and two-dimensional differential equations. We also include several lectures on neural and gene networks, cellular automata, and fractals.

Although this text is written at a level appropriate to first- and second-year undergraduates, most of the material dealing with nonlinear finite-difference and differential equations and time-series analysis is not presented in standard undergraduate or graduate curricula in the physical sciences or mathematics. This book might well be used as a source for supplementary material for traditional courses in advanced calculus, differential equations, and mathematical methods in physical sciences. The link between dynamics and time series analysis can make this book useful to statisticians or signal processing engineers interested in a new perspective on their subject and in an introduction to the research literature.

Over the years, a number of teaching assistants have contributed to the development of this material and the education of the students. Particular thanks go to Carl Graves, David Larocque, Wanzhen Zeng, Marc Courtemanche, Hiroyuki Ito, and Gil Bub. We also thank Michael Broide, Scott Greenwald, Frank Witkowski, Bob Devaney, Michael Shlesinger, Jim Crutchfield, Melanie Mitchell, Michael Frame, Jerry Marsden, and the students of McGill University Biology 309 for their many corrections and suggestions. We thank André Duchastel for his careful redrawing of many of the figures reproduced from other sources. Finally, we thank Jerry Lyons, Liesl Gibson, Karen Kosztolnyik, and Kristen Cassereau for their excellent editorial assistance and help in the final stages of preparation of this book.

McGill University has provided an ideal environment to carry out research and to teach. Our colleagues and chairmen have provided encouragement in many ways. We would like to thank in particular, J. Milic-Emili, K. Krjnevic, D. Goltzman, A. Shrier, M. R. Guevara, and M. C. Mackey. The financial support of the Natural Sciences Engineering and Research Council (Canada), the Medical Research Council (Canada), the Canadian Heart and Stroke Association has enabled us to carry out research that is reflected in the text. Finally, Leon Glass thanks the John Simon Guggenheim Memorial Foundation for Fellowship support during the final stages of the preparation of this text.

We are making available various electronic extensions to this book, including additional exercises, solutions, and computer materials. For information, please contact [understanding@cnd.mcgill.ca](mailto:understanding@cnd.mcgill.ca).

February 1995

Daniel Kaplan  
[danny@cnd.mcgill.ca](mailto:danny@cnd.mcgill.ca)  
Leon Glass  
[glass@cnd.mcgill.ca](mailto:glass@cnd.mcgill.ca)

# About the Authors

Daniel Kaplan specializes in the analysis of data using techniques motivated by nonlinear dynamics. His primary interest is in the interpretation of irregular physiological rhythms, but the methods he has developed have been used in geophysics, economics, marine ecology, and other fields. He joined McGill in 1991, after receiving his Ph.D from Harvard University and working at MIT. His undergraduate studies were completed at Swarthmore College. He has worked with several instrumentation companies to develop novel types of medical monitors.

Leon Glass is one of the pioneers of what has come to be called chaos theory, specializing in applications to medicine and biology. He has worked in areas as diverse as physical chemistry, visual perception, and cardiology, and is one of the originators of the concept of “dynamical disease.” He has been a professor at McGill University in Montreal since 1975, and has worked at the University of Rochester, the University of California in San Diego, and Harvard University. He earned his Ph.D. at the University of Chicago and did postdoctoral work at the University of Edinburgh and the University of Chicago.

# Contents

---

## **PREFACE**

**vii**

---

## **ABOUT THE AUTHORS**

**xiii**

---

## **1    FINITE-DIFFERENCE EQUATIONS**

**1**

1.1	A Mythical Field	1
1.2	The Linear Finite-Difference Equation	2
1.3	Methods of Iteration	6
1.4	Nonlinear Finite-Difference Equations	8
1.5	Steady States and Their Stability	12
1.6	Cycles and Their Stability	20
1.7	Chaos	27
1.8	Quasiperiodicity	33

1	Chaos in Periodically Stimulated Heart Cells	37
	Sources and Notes	41
	Exercises	42
	Computer Projects	51
<b>2</b>	<b>BOOLEAN NETWORKS AND CELLULAR AUTOMATA</b>	<b>55</b>
2.1	Elements and Networks	56
2.2	Boolean Variables, Functions, and Networks	58
2	A Lambda Bacteriophage Model	64
3	Locomotion in Salamanders	70
2.3	Boolean Functions and Biochemistry	73
2.4	Random Boolean Networks	77
2.5	Cellular Automata	79
4	Spiral Waves in Chemistry and Biology	88
2.6	Advanced Topic: Evolution and Computation	91
	Sources and Notes	94
	Exercises	96
	Computer Projects	101
<b>3</b>	<b>SELF-SIMILARITY AND FRACTAL GEOMETRY</b>	<b>105</b>
3.1	Describing a Tree	106
3.2	Fractals	109
3.3	Dimension	111
5	The Box-Counting Dimension	115
3.4	Statistical Self-Similarity	116
6	Self-Similarity in Time	117
3.5	Fractals and Dynamics	121
7	Random Walks and Lévy Walks	126
8	Fractal Growth	137
	Sources and Notes	141
	Exercises	142
	Computer Projects	143

## 4 ONE-DIMENSIONAL DIFFERENTIAL EQUATIONS 147

---

- 4.1 Basic Definitions 148
- 4.2 Growth and Decay 149
  - 9 Traffic on the Internet 156
  - 10 Open Time Histograms in Patch Clamp Experiments 158
  - 11 Gompertz Growth of Tumors 163
- 4.3 Multiple Fixed Points 164
- 4.4 Geometrical Analysis of One-Dimensional Nonlinear Ordinary Differential Equations 166
- 4.5 Algebraic Analysis of Fixed Points 168
- 4.6 Differential Equations versus Finite-Difference Equations 172
- 4.7 Differential Equations with Inputs 174
  - 12 Heart Rate Response to Sinusoid Inputs 182
- 4.8 Advanced Topic: Time Delays and Chaos 183
  - 13 Nicholson's Blowflies 186
- Sources and Notes 188
- Exercises 189
- Computer Projects 205

## 5 TWO-DIMENSIONAL DIFFERENTIAL EQUATIONS 209

---

- 5.1 The Harmonic Oscillator 209
- 5.2 Solutions, Trajectories, and Flows 211
- 5.3 The Two-Dimensional Linear Ordinary Differential Equation 213
- 5.4 Coupled First-Order Linear Equations 219
  - 14 Metastasis of Malignant Tumors 221
- 5.5 The Phase Plane 226
- 5.6 Local Stability Analysis of Two-Dimensional, Nonlinear Differential Equations 230
- 5.7 Limit Cycles and the van der Pol Oscillator 240
- 5.8 Finding Solutions to Nonlinear Differential Equations 244
  - 15 Action Potentials in Nerve Cells 245
- 5.9 Advanced Topic: Dynamics in Three or More Dimensions 248

5.10	Advanced Topic: Poincaré Index Theorem	253
	Sources and Notes	260
	Exercises	260
	Computer Projects	275

---

## 6 TIME-SERIES ANALYSIS 279

6.1	Starting with Data	279
6.2	Dynamics, Measurements, and Noise	280
	16 Fluctuations in Marine Populations	281
6.3	The Mean and Standard Deviation	286
6.4	Linear Correlations	291
6.5	Power Spectrum Analysis	298
	17 Daily Oscillations in Zooplankton	300
6.6	Nonlinear Dynamics and Data Analysis	303
	18 Reconstructing Nerve Cell Dynamics	304
6.7	Characterizing Chaos	314
	19 Predicting the Next Ice Age	330
6.8	Detecting Chaos and Nonlinearity	338
6.9	Algorithms and Answers	347
	Sources and Notes	348
	Exercises	349
	Computer Projects	353

---

## APPENDIX A A MULTI-FUNCTIONAL APPENDIX 359

A.1	The Straight Line	361
A.2	The Quadratic Function	362
A.3	The Cubic and Higher-Order Polynomials	362
A.4	The Exponential Function	363
A.5	Sigmoidal Functions	364
A.6	The Sine and Cosine Functions	367
A.7	The Gaussian (or "Normal") Distribution	368
A.8	The Ellipse	370
A.9	The Hyperbola	371
	Exercises	371

**APPENDIX B A NOTE ON COMPUTER  
NOTATION****381****SOLUTIONS TO SELECTED EXERCISES****385****BIBLIOGRAPHY****401****INDEX****409**



# Finite-Difference Equations

## 1.1 A MYTHICAL FIELD

---

Imagine that a graduate student goes to a meadow on the first day of May, walks through the meadow waving a fly net, and counts the number of flies caught in the net. She repeats this ritual for several years, following up on the work of previous graduate students. The resulting measurements might look like the graph shown in Figure 1.1. The graduate student notes the variability in her measurements and wants to find out if they contain any important biological information.

Several different approaches could be taken to study the data. The student could do statistical analyses of the data to calculate the mean value or to detect long-term trends. She could also try to develop a detailed and realistic model of the ecosystem, taking into account such factors as weather, predators, and the fly populations in previous years. Or she could construct a simplified theoretical model for fly population density.

Sticking to what she knows, the student decides to model the population variability in terms of actual measurements. The number of flies in one summer