Graduate Texts in Mathematics

Real and Functional Analysis

Third Edition

实分析和泛函分析

第3版

Serge Lang

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Third Edition

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Foreword

This book is meant as a text for a first year graduate course in analysis. Any standard course in undergraduate analysis will constitute sufficient preparation for its understanding, for instance, my *Undergraduate Analysis*. I assume that the reader is acquainted with notions of uniform convergence and the like.

In this third edition, I have reorganized the book by covering integration before functional analysis. Such a rearrangement fits the way courses are taught in all the places I know of. I have added a number of examples and exercises, as well as some material about integration on the real line (e.g. on Dirac sequence approximation and on Fourier analysis), and some material on functional analysis (e.g. the theory of the Gelfand transform in Chapter XVI). These upgrade previous exercises to sections in the text.

In a sense, the subject matter covers the same topics as elementary calculus, viz. linear algebra, differentiation and integration. This time, however, these subjects are treated in a manner suitable for the training of professionals, i.e. people who will use the tools in further investigations, be it in mathematics, or physics, or what have you.

In the first part, we begin with point set topology, essential for all analysis, and we cover the most important results.

I am selective here, since this part is regarded as a tool, especially Chapters I and II. Many results are easy, and are less essential than those in the text. They have been given in exercises, which are designed to acquire facility in routine techniques and to give flexibility for those who want to cover some of them at greater length. The point set topology simply deals with the basic notions of continuity, open and closed sets, connectedness, compactness, and continuous functions. The chapter concerning continuous functions on compact sets properly emphasizes results which already mix analysis and uniform convergence with the language of point set topology.

In the second part, Chapters IV and V, we describe briefly the two basic linear spaces of analysis, namely Banach spaces and Hilbert spaces.

The next part deals extensively with integration.

We begin with the development of the integral. The fashion has been to emphasize positivity and ordering properties (increasing and decreasing sequences). I find this excessive. The treatment given here attempts to give a proper balance between L^1-convergence and positivity. For more detailed comments, see the introduction to Part Three and Chapter VI.

The chapters on applications of integration and distributions provide concrete examples and choices for leading the course in other directions, at the taste of the lecturer. The general theory of integration in measured spaces (with respect to a given positive measure) alternates with chapters giving specific results of integration on euclidean spaces or the real line. Neither is slighted at the expense of the other. In this third edition, I have added some material on functions of bounded variation, and I have emphasized convolutions and the approximation by Dirac sequences or families even more than in the previous editions, for instance, in Chapter VIII, §2.

For want of a better place, the calculus (with values in a Banach space) now occurs as a separate part after dealing with integration, and before the functional analysis.

The differential calculus is done because at best, most people will only be acquainted with it only in euclidean space, and incompletely at that. More importantly, the calculus in Banach spaces has acquired considerable importance in the last two decades, because of many applications like Morse theory, the calculus of variations, and the Nash–Moser implicit mapping theorem, which lies even further in this direction since one has to deal with more general spaces than Banach spaces. These results pertain to the geometry of function spaces. Cf. the exercises of Chapter XIV for simpler applications.

The next part deals with functional analysis. The purpose here is twofold. We place the linear algebra in an infinite dimensional setting where continuity assumptions are made on the linear maps, and we show how one can "linearize" a problem by taking derivatives, again in a setting where the theory can be applied to function spaces. This part includes several major spectral theorems of analysis, showing how we can extend to the infinite dimensional case certain results of finite dimensional linear algebra. The compact and Fredholm operators have applications to integral operators and partial differential elliptic operators (e.g. in papers of Atiyah–Singer and Atiyah–Bott).

Chapters XIX and XXIX, on unbounded hermitian operators, combine both the linear algebra and integration theory in the study of such operators. One may view the treatment of spectral measures as providing an example of general integration theory on locally compact spaces, whereby a measure is obtained from a functional on the space of continuous functions with compact support.

I find it appropriate to introduce students to differentiable manifolds during this first year graduate analysis course, not only because these objects are of interest to differential geometers or differential topologists, but because global analysis on manifolds has come into its own, both in its integral and differential aspects. It is therefore desirable to integrate manifolds in analysis courses, and I have done this in the last part, which may also be viewed as providing a good application of integration theory.

A number of examples are given in the text but many interesting examples are also given in the exercises (for instance, explicit formulas for approximations whose existence one knows abstractly by the Weierstrass-Stone theorem; integral operators of various kinds; etc). The exercises should be viewed as an integral part of the book. Note that Chapters XIX and XX, giving the spectral measure, can be viewed as providing an example for many notions which have been discussed previously: operators in Hilbert space, measures, and convolutions. At the same time, these results lead directly into the real analysis of the working mathematician.

As usual, I have avoided as far as possible building long chains of logical interdependence, and have made chapters as logically independent as possible, so that courses which run rapidly through certain chapters, omitting some material, can cover later chapters without being logically inconvenienced.

The present book can be used for a two-semester course, omitting some material. I hope I have given a suitable overview of the basic tools of analysis. There might be some reason to include other topics, such as the basic theorems concerning elliptic operators. I have omitted this topic and some others, partly because the appendices to my $SL_2(\mathbf{R})$ constitutes a sub-book which contains these topics, and partly because there is no time to cover them in the basic one year course addressed to graduate students.

The present book can also be used as a reference for basic analysis, since it offers the reader the opportunity to select various topics without reading the entire book. The subject matter is organized so that it makes the topics available to as wide an audience as possible.

There are many very good books in intermediate analysis, and interesting research papers, which can be read immediately after the present course. A partial list is given in the Bibliography. In fact, the determination of the material included in this *Real and Functional Analysis* has been greatly motivated by the existence of these papers and books, and by the need to provide the necessary background for them.

Finally, I thank all those people who have made valuable comments and corrections, especially Keith Conrad, Martin Mohlenkamp, Takesi Yamanaka, and Stephen Chiappari, who reviewed the book for Springer-Verlag.

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Contents

PART ONE

General Topology

CHAPTER I

Sets

I, §1. SOME BASIC TERMINOLOGY

We assume that the reader understands the meaning of the word "set", and in this chapter, summarize briefly the basic properties of sets and operations between sets. We denote the empty set by $\varnothing$. A subset S' of S is said to be **proper** if $S' \neq S$. We write $S' \subset S$ or $S \supset S'$ to denote the fact that S' is a subset of S.

Let S, T be sets. A **mapping** or **map** $f: T \to S$ is an association which to each element $x \in T$ associates an element of S, denoted by $f(x)$, and called the **value** of f at x, or the **image** of x under f. If T' is a subset of T, we denote by $f(T')$ the subset of S consisting of all elements $f(x)$ for $x \in T'$. The association of $f(x)$ to x is denoted by the special arrow

$$x \mapsto f(x).$$

We usually reserve the word function for a mapping whose values are in the real or complex numbers. The **characteristic function** of a subset S' of S is the function χ such that $\chi(x) = 1$ if $x \in S'$ and $\chi(x) = 0$ if $x \notin S'$. We often write $\chi_{S'}$ for this function.

Let X, Y be sets. A map $f: X \to Y$ is said to be **injective** if for all x, $x' \in X$ with $x \neq x'$ we have $f(x) \neq f(x')$. We say that f is **surjective** if $f(X) = Y$, i.e. if the image of f is all of Y. We say that f is **bijective** if it is both injective and surjective. As usual, one should index a map f by its set of arrival and set of departure to have absolutely correct notation, but this is too clumsy, and the context is supposed to make it clear what these sets are. For instance, let $\mathbf{R}$ denote the real numbers, and $\mathbf{R}'$ the real numbers $\geqq 0$. The map

$$f: \mathbf{R} \to \mathbf{R}$$

given by $x \mapsto x^2$ is not surjective, but the map

$$f: \mathbf{R} \to \mathbf{R}'$$

given by the same formula is surjective.

If $f: X \to Y$ is a map and S a subset of X, we denote by

$$f|S$$

the restriction of f to S, namely the map f viewed as a map defined only on S. For instance, if $f: \mathbf{R} \to \mathbf{R}'$ is the map $x \mapsto x^2$, then f is not injective, but $f|\mathbf{R}'$ is injective. We often let $f_S = f\chi_S$ be the function equal to f on S and 0 outside S.

A composite of injective maps is injective, and a composite of surjective maps is surjective. Hence a composite of bijective maps is bijective.

We denote by $\mathbf{Q}$, $\mathbf{Z}$ the sets of rational numbers and integers respectively. We denote by $\mathbf{Z}^+$ the set of positive integers (integers > 0), and similarly by $\mathbf{R}^+$ the set of positive reals. We denote by $\mathbf{N}$ the set of natural numbers (integers $\geqq 0$), and by $\mathbf{C}$ the complex numbers. A mapping into $\mathbf{R}$ or $\mathbf{C}$ will be called a **function**.

Let S and I be sets. By a **family of elements of** S, **indexed by** I, one means simply a map $f: I \to S$. However, when we speak of a family, we write $f(i)$ as f_i, and also use the notation $\{f_i\}_{i \in I}$ to denote the family.

Example 1. Let S be the set consisting of the single element 3. Let $I = \{1, \ldots, n\}$ be the set of integers from 1 to n. A family of elements of S, indexed by I, can then be written $\{a_i\}_{i=1,\ldots,n}$ with each $a_i = 3$. Note that a family is different from a subset. The same element of S may receive distinct indices.

A family of elements of a set S indexed by positive integers, or nonnegative integers, is also called a **sequence**.

Example 2. A sequence of real numbers is written frequently in the form

$$\{x_1, x_2, \ldots\} \qquad \text{or} \qquad \{x_n\}_{n \geqq 1}$$

and stands for the map $f: \mathbf{Z}^+ \to \mathbf{R}$ such that $f(i) = x_i$. As before, note that a sequence can have all its elements equal to each other, that is

$$\{1, 1, 1, \ldots\}$$

is a sequence of integers, with $x_i = 1$ for each $i \in \mathbf{Z}^+$.