FUNDAMENTAL THEORY

BY

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EDITOR'S PREFACE

AFTER the death of Sir Arthur Eddington in November 1944, Miss Eddington and the Syndics of the Cambridge University Press invited me to supervise the publication of the manuscript of the present volume. I have followed the text without alteration except in correcting obvious slips of the pen, making slight changes to remove obscurities, and supplying references which Eddington had left blank. I have added a few additional notes and have constructed an Index.

By way of general introduction, I may say that the work is complete in itself, and practically replaces all the author's previous writings on his theory of the constants of Nature. Chapters I–V follow closely the treatment adopted in his Dublin lectures of 1943, and Chapters VI–VIII are devoted to the sedenion analysis which had been expounded in his *Relativity Theory of Protons and Electrons* of 1936; but the rest of the book is chiefly new matter, and contains developments of outstanding power and interest. Those who desire a preliminary glance at the results may be advised to turn to:

(i) The table on page 66, which gives the values of the microscopic constants as calculated by Eddington's theory, compared with the observed values.

(ii) The similar table of molar and nuclear constants on page 105.

(iii) The first list of achievements of the theory in nuclear physics given on page 211: the numerical comparisons will be found in the separate sections of Chapter IX, Eddington's intention having been to collect them, together with the discoveries of Chapters X–XII, in a table in the part of the book which he did not live to complete.

(iv) The results for the magnetic moments of the hydrogen atom and the neutron on pages 249 and 251.

For a somewhat fuller introductory account of the theory, reference may be made to an article in the *Mathematical Gazette*, **29** (October 1945), pp. 137–44.

Professor E. T. Copson, of University College, Dundee, in the University of St Andrews, and Professor George Temple, F.R.S., Head of the mathematics department in King's College, London, have most kindly read the proof-sheets with me. I wish also to acknowledge gratefully the help given by the Staff of the Cambridge University Press.

EDMUND T. WHITTAKER

CONTENTS

Chapter I

THE UNCERTAINTY OF THE REFERENCE FRAME

1. The uncertainty of the origin

The quantities occurring in the equations of mathematical physics relate partly to physical objects and events and partly to a mathematical framework introduced for purposes of reference. Both relativity theory and quantum theory stress the distinction between *observables*, i.e. quantities which could be ascertained by a specified observational procedure, and *unobservables*, i.e. quantities depending partly on the auxiliary mathematical frame which cannot be the subject of actual observation. Unobservables are used to systematise description and facilitate calculation, but they are eliminated in the final steps of the calculation of observationally verifiable results.

In classifying quantities as observable or unobservable I shall follow current usage rather than the most literal meaning of the terms. Rightly or wrongly, modern physics is not over-scrupulous in postulating measurements of a highly impracticable kind. But although measurements are often very much idealised, it is recognised that idealisation must stop short of actual self-contradiction. Relativity theory and quantum theory have each laid down conditions of observability which are certainly necessary if self-contradiction is to be avoided; and the term 'observable' will here be understood to mean that the quantity satisfies both tests.[a]

Relativity theory begins with a denial of absolute motion. An observed velocity $d\xi/dt$ of a physical entity is necessarily relative to another physical entity. Likewise an observed coordinate ξ is a relative coordinate of two physical entities.

Quantum theory insists that the connection of a physical entity with the geometrical frame of coordinates is governed by Heisenberg's uncertainty principle. A particle is not exactly locatable at a point in the geometrical frame, or in four dimensions as a world-line. It can only be assigned a probability distribution of position and velocity.

In modern physics these two principles of observability have been applied separately with very far-reaching results; but they have seldom been applied in combination even by those who profess to be developing a relativistic quantum theory. The combined principle is that *a coordinate ξ is observable only if it is a relative coordinate of two entities both of which have uncertainty of position and momentum in the geometrical frame.*

The same considerations apply to momenta and other observables. An observable is always a statistic of a double probability distribution.

The essential point is that an observable coordinate is measured, not from an abstract mathematical point as origin, but from something which is involved physically in the operation which furnishes the measure. Being involved physically it experiences those incalculable reactions which limit the precision of our knowledge in the way laid down by Heisenberg's principle. We must therefore distinguish between the 'physical origin' from which an observable coordinate is measured, and the 'geometrical origin' of the auxiliary mathematical frame. The latter, as already stated, is

[a] Observables and measurables will be fully treated in Chapter XIII, and a logically satisfactory definition will then be given.

eliminated in the final calculation of observationally verifiable results; being therefore aloof from the rough-and-tumble of observational inquisition, it has a sharpness of definition which contrasts with the blurring of all physical landmarks by probability scatter.

Consider a system of particles with coordinates x_r, y_r, z_r $(r = 1, 2, 3, \ldots)$ in the geometrical frame. These coordinates are unobservables. To obtain physical, i.e. observable, coordinates we must substitute for the geometrical origin an actual particle or its equivalent, e.g. the centroid of a set of particles. Let the geometrical coordinates of the physical origin be x_0, y_0, z_0; these also are unobservable. But the relative coordinates

$$\xi_r, \eta_r, \zeta_r = x_r - x_0, y_r - y_0, z_r - z_0 \tag{1·1}$$

are observables.

Nominally the exact value of ξ_r could be found by observation, at the expense of infinite uncertainty of the conjugate momentum. Such a measurement would be scientifically useless, since the coordinate would instantly become uncertain again; it is the measurement of a careless experimenter who destroys his specimen by handling it too roughly. Thus our knowledge of ξ_r, η_r, ζ_r at any time is described by a probability distribution. A measurement of ξ_r will give a value taken at random from the pre-existing[a] probability distribution of ξ_r; or equivalently it will give the distance from a random point in the probability distribution of x_0 to a random point in the probability distribution of x_r.

The transformation of coordinates from x_r, y_r, z_r to ξ_r, η_r, ζ_r is a change from an origin fixed in the geometrical frame to an origin with a probability scatter in that frame. It will be necessary later to make a special study of this type of transformation which, of course, is beyond the scope of the ordinary theory of coordinate transformations. In particular, we shall obtain formulae for transforming a probability distribution of physical coordinates into a probability distribution of geometrical coordinates or vice versa, and very much simpler formulae for transforming the probability distribution of the conjugate momenta (§§ 37, 38). But to carry out these transformations it is necessary to know the distribution function $f(x_0, y_0, z_0)$ of the coordinates of the physical origin. This function cannot be found observationally, because x_0, y_0, z_0 are unobservable.

The coordinates postulated in the dynamical equations of wave mechanics must be measured from a physical origin, since they and their conjugate momenta are assumed to be observables, being in fact the typical observables of quantum theory. It will be recalled that the wave-packets, whose propagation and diffusion are studied in wave mechanics, are created by our observational measurements—or more strictly by our becoming aware of the results of measurements and assessing the probability accordingly—so that it is essential to distinguish the variates in which these concentrations of probability can occur.

Thus in some, if not all, of the fundamental equations of quantum theory the coordinates are measured from a physical origin. The urgent question arises: What is this origin, and what distribution function $f(x_0, y_0, z_0)$ has been assumed for it? Writers

[a] After the measurement the information which it furnishes is used to reassess the probability. The probability distribution therefore changes discontinuously at the moment when the observer becomes aware of the result of the measurement. Attention will be paid to this point in § 35, where a very important distinction between 'structural theory' and 'predictive theory' is introduced.

on quantum theory give no hint as to the physical origin they are employing. But their equations can only be valid for some particular origin, since they are not of a form which would be invariant for arbitrary changes of f.

2. The physical origin

The centroid of a large number of particles has the important statistical property that (subject to certain conditions which are ordinarily fulfilled) the form of its probability distribution does not depend on the law of probability distribution of the individual particles. The mean of a large number of uncorrelated variates x_r has a Gaussian distribution whatever, within reason, may be the distribution law of the individual x_r.

Thus if we employ the centroid of a large number of particles as our physical origin, we have the immense advantage of starting with an *a priori* knowledge of the distribution of its geometrical coordinates x_0, y_0, z_0, complete except for the one disposable constant in the Gaussian law. The distribution function of x_0 is then

$$f(x_0) = (2\pi\sigma^2)^{-\frac{1}{2}} e^{-x_0^2/2\sigma^2}.$$

If we impose the condition that the distribution of the particles has spherical symmetry, the formula is extended to three dimensions and becomes

$$f(x_0, y_0, z_0) = (2\pi\sigma^2)^{-\frac{3}{2}} e^{-(x_0^2+y_0^2+z_0^2)/2\sigma^2}. \tag{2·1}$$

The standard deviation σ, which is left to be determined later, will be called the *uncertainty constant* of the physical reference frame.

Although the centroid is not directly indicated by a physical landmark, it is admissible as a physical origin. Formally the observational procedure would be to measure the coordinate ξ_{rs} of the rth particle from each of the other particles in turn, and take the mean $\xi_r = \overline{\xi}_{rs}$; the mean can be treated as equivalent to a single measurement from a mean particle at x_0.

Throughout this book we shall employ a physical origin related to the geometrical origin by the Gaussian distribution function (2·1), which is defined observationally as the centroid of a system of n particles with a spherically symmetrical but otherwise unrestricted probability distribution. The number n is always understood to be very large. Sometimes this assemblage will be the principal subject of investigation; but, if not, it is in the background, forming the standard environment (§ 7) of the small object-system that is being particularly studied.

The laws and constants that we derive are valid only on the understanding that the measurements concerned in them are referred to the physical frame defined by (2·1). Since writers on quantum theory leave us to guess what frame they are using, there is no guarantee that our frame will turn out to be the one in which the current quantum equations are valid. We are under no obligation to prove this identity in advance; but as a matter of practical expediency it is very desirable that the frames should agree. By making an early junction with current quantum theory we are saved an overwhelming amount of labour, because we can then take over unchanged all the specialised investigations needed to complete the application of our results to practical experiments; and it is therefore good policy to avoid unnecessary differences of form and definitions. The carrying out of this policy involves a good deal of 'intelligent anticipation', and steps which determine the form of the theory often have to be justified from this point of view. The reader interested in logical rigour should bear in mind that the development of the theory turns partly on strict deduction and partly on ultimate saving of labour. The former part requires proof, the latter part success.

Fortunately, we can foresee that current quantum theory must be based on a physical origin which is the centroid of a large number of particles. For if it were otherwise, the equations could be of no practical use. Since x_0, y_0, z_0 are unobservable, there is no way of determining $f(x_0, y_0, z_0)$ by observation. It is only when we have theoretical information, such as that furnished by 'the law of large numbers', that we can associate a definite form of f with an observationally defined point. Thus, if quantum theory postulates a non-Gaussian form of f, it is impossible to recognise observationally the measured coordinates to which it applies, and there is no means of connecting its predictions with actual experiment. This is a *reductio ad absurdum*, because there is no doubt as to the general agreement of current quantum theory with experiment.

The foregoing may be described as the problem of 'anchoring' an ideal mathematical frame in the world of observational measurement. Anchoring is made possible by the statistical cancelling of fluctuations in large assemblages; and the small residual fluctuation that remains is necessarily Gaussian.

Starting with an abstract geometrical coordinate frame, we step over from pure geometry into physics by introducing a physical coordinate frame whose origin has the probability distribution (2·1) relative to the geometrical origin. We shall find that the standard deviation σ of this distribution *puts the scale into* the physical frame and everything constructed in the physical frame, whether it be a nucleus, an atom, a crystal or the whole extent of physical space. The main problem in this book is to investigate the way in which the extensions of these various structures are related to σ, and to evaluate the numerical ratios for some of the simpler structures.

In stepping over from the geometrical to the physical frame we appear to have freedom of choice of σ. But the freedom is illusory, because σ can only be measured in terms of the extensions of physical structures whose scale it has itself determined. To double σ would double all linear constants such as the wave-lengths of the hydrogen spectrum; thus the measure of σ in terms of the wave-length of the H_α line as unit would remain unaltered.

3. The Bernoulli fluctuation

Consider a very large number of particles N which all have the same probability distribution of coordinates. Let V_0 be a volume, fixed in the geometrical frame, extensive enough to include a large number of them. Each particle has the same probability p of being within V_0, and the mean or expectation number in V_0 is $n_0 = pN$. Let the actual number in V_0 be n, and set

$$n = n_0 + y.$$

Then, by James Bernoulli's theorem, the fluctuation y has the distribution law

$$f_N(y) = \{2\pi n_0(1-n_0/N)\}^{-\frac{1}{2}} e^{-y^2/2n_0(1-n_0/N)}. \tag{3·11}$$

If $N/n_0 \to \infty$, this becomes

$$f_\infty(y) = (2\pi n_0)^{-\frac{1}{2}} e^{-y^2/2n_0}. \tag{3·12}$$

Both distributions are Gaussian, and their standard deviations are $(n_0 - n_0^2/N)^{\frac{1}{2}}$ and $n_0^{\frac{1}{2}}$. Hence (3·12) can be obtained by compounding with (3·11) an independent Gaussian fluctuation with standard deviation $(n_0^2/N)^{\frac{1}{2}}$ and distribution law

$$f_e(y) = (2\pi n_0^2/N)^{-\frac{1}{2}}\, e^{-Ny^2/2n_0^2}. \tag{3·13}$$

Let $\zeta = y/n_0$, so that $$n = n_0(1+\zeta). \tag{3·2}$$

Then the distribution law of ζ corresponding to $f_e(y)$ is

$$g_e(\zeta) = (2\pi/N)^{-\frac{1}{2}}\, e^{-\frac{1}{2}N\zeta^2}. \tag{3·3}$$

We thus resolve the Bernoulli fluctuation into two independent Gaussian fluctuations, namely, an 'ordinary fluctuation' (3·12) arising from the finiteness of n_0 and an 'extraordinary fluctuation' (3·3) arising from the finiteness of N. The extraordinary fluctuation is to be combined negatively, so as to give a total fluctuation less than the ordinary fluctuation.

We shall apply this analysis to a system of particles which is in self-equilibrium, so that the probability distribution is steady. According to relativity theory the only distribution of matter which can be in self-equilibrium is a uniform distribution filling a hyperspherical space. This is the well-known 'Einstein universe'. The hyperspherical (or, as it is commonly called, spherical) space has finite volume; so that N/n_0 is finite. The infinite Euclidean space of classical theory corresponds to the limit when $N/n_0 \to \infty$ and the extraordinary fluctuation vanishes. Thus, in passing from classical to relativity theory by taking N finite, two changes are made: the space becomes curved, and an extraordinary fluctuation is introduced. These, however, are not two changes but one. *We are going to show that the space curvature is simply a way of taking into account the extraordinary fluctuation.*

Henceforth we shall deal with the extraordinary fluctuation alone. (The ordinary fluctuation, being common to relativity theory and classical theory, requires no special attention.) Denoting the particle density n/V_0 by s, the fluctuation changes an exact particle density s_0 into a slightly uncertain density

$$s = s_0(1+\zeta). \tag{3·41}$$

Instead of considering an uncertain number of particles n in a fixed volume V_0, we can consider an exact number of particles n_0 and transfer the uncertainty to the containing volume V, where $n/V_0 = n_0/V$. Setting

$$V = V_0/(1+\epsilon)^3, \tag{3·42}$$

the uncertainty is now contained in a linear scale factor $1+\epsilon$.

The distribution function $g_e(\zeta)$ can be transformed into a distribution function of ϵ. If we had to transform a distribution over discrete values of ζ into a distribution over corresponding values of ϵ, the relation would be $(1+\zeta) = (1+\epsilon)^3$. But, in transforming a continuous distribution function, discrete values are replaced by constant ranges, and we have to insert a factor proportional to $d\epsilon/d\zeta$ to transform constant ranges of ϵ into the non-constant ranges of ϵ which correspond to constant ranges of ζ. The relation is therefore

$$(1+\zeta)\,d\zeta = \text{constant} \times (1+\epsilon)^3\, d\epsilon,$$

which gives on integration $$(1+\zeta)^2 = (1+\epsilon)^4. \tag{3·43}$$

For the distribution function $g_e(\zeta)$ the values of ζ which have sensible probability are of order $N^{-\frac{1}{2}}$, and are therefore extremely small—actually about 10^{-39}. Hence (3·43) becomes with ample approximation $\zeta = 2\epsilon$. By (3·3), the standard deviation of ζ is $N^{-\frac{1}{2}}$; hence the standard deviation of ϵ is

$$\sigma_\epsilon = 1/2\sqrt{N}. \tag{3·5}$$

The extraordinary fluctuation of the particle density can therefore be represented by a scale fluctuation with the standard deviation (3·5).

The geometrical frame is our standard of fixity when we speak of the uncertainties of physical quantities; and the ideal exact scale $\epsilon = 0$ is the scale on which the geometrical coordinates are measured. In order to take account of the extraordinary fluctuation as a scale uncertainty, we must introduce the uncertain scale $1+\epsilon$ in the system of the physical coordinates ξ, η, ζ. Considering a point distant r from the origin, the difference $x_0, y_0, z_0 = \xi - x, \eta - y, \zeta - z$ between its physical and geometrical coordinates will now consist of

(*a*) a fluctuation with standard deviation σ in all directions, due to the uncertainty of the position of the physical origin, and

(*b*) a fluctuation with standard deviation $\sigma_\epsilon r$ in the radial direction only, due to the uncertainty of the scale of measurement of r.

Remembering that the extraordinary fluctuation represented by (*b*) is to be combined negatively with other sources of fluctuation, the resultant standard deviation is

$$\text{radial } (\sigma^2 - \sigma_\epsilon^2 r^2)^{\frac{1}{2}}, \qquad \text{transverse } \sigma. \tag{3·6}$$

We shall call (3·6) the *local uncertainty* of the physical reference frame. It has been derived as a combination of the uncertainty of a distant origin with the uncertainty of scale; but it can be described more compactly as the uncertainty of a local physical origin relative to a local geometrical origin. We could, by making a local coordinate transformation, introduce 'natural coordinates' such that the local uncertainty in all directions is restored to the original value σ; these coordinates are applicable so long as the distance r from the local origin is small enough for $\sigma_\epsilon r$ to be neglected. Independently of coordinate systems, *the local uncertainty in a given direction defines an extension which might be adopted as the unit for measuring lengths in that direction in that locality.* We shall call this the σ-system of defining lengths, or briefly 'the σ-metric'.

Let ds be the length, reckoned in σ-metric, of a line-element dr, $r\,d\theta$, $r\sin\theta\,d\phi$. By (3·6) the lengths of radial and transverse elements are proportional to $dr/(\sigma^2 - \sigma_\epsilon^2 r^2)^{\frac{1}{2}}$, $r\,d\theta/\sigma$, $r\sin\theta\,d\phi/\sigma$; so that the general formula is

$$ds^2 = \frac{dr^2}{1-(\sigma_\epsilon^2/\sigma^2)\,r^2} + r^2 d\theta^2 + r^2\sin^2\theta\,d\phi^2. \tag{3·7}$$

This is the well-known formula for the line-element in spherical space of radius $R_0 = \sigma/\sigma_\epsilon$. Hence by (3·5)

$$\sigma = R_0/2\sqrt{N}. \tag{3·8}$$

It will appear in the next section that the σ-metric is the recognised metric in physical theory; so that we have in this way reached the usual description of space (occupied by a steady distribution of particles) as spherical, and have found the equation (3·8) which determines the uncertainty constant σ in terms of the cosmological constants R_0, N.

The scale uncertainty is naturally interpreted as the result of measuring with a standard whose actual extension (in the geometrical coordinate frame) is uncertain to this extent. A standard whose uncertainty is no more than 1 part in 10^{39} is very much idealised; but there is no self-contradiction in assigning to a physical system characteristics supposed to have been measured by such a standard. If, on the other hand, the standard is supposed to be exact, it is contradictory to suppose that anything has been measured with it.

If a measured distance r has been determined with a standard which has a fluctuation σ_ϵ, the standard deviation of r cannot be less than $\sigma_\epsilon r$. We have applied the standard deviation $\sigma_\epsilon r$ negatively; this means that we have corrected the whole observed fluctuation of r for the part attributable to uncertainty of the standard. We first represent the observational measures, without any correction, in flat physical space. Recognising that the probability scatter is partly due to fluctuation of the standard employed (it being impossible to make measures at all unless the standard is inexact), we desire to eliminate this part so as to obtain what we should regard as the true distribution corresponding to an exact standard. This elimination changes the σ-metric so that the flat space is transformed into the spherical space (3·7). Thus the cosmical curvature replaces the fluctuation of the standard; and when we use the relativity representation of the universe in spherical space the scale is to be treated as exact.

This elimination of σ_ϵ is statistical, so that the curvature representation is suitable for molar physics, which is concerned with statistical averages of large numbers of particles. But a merely statistical elimination is not good enough for microscopic physics. Consequently in quantum theory we shall not use the curvature representation. We shall revert to flat space, and take account of scale fluctuation in another way (§ 24).

4. The standard of length[a]

In order to make it clear that the σ-metric is the recognised metric in relativity theory, quantum theory and practical metrology, we consider the conditions that must be fulfilled by an ultimate standard of length. The Paris metre is not accepted as an ultimate standard; the mere fact that anxiety is felt as to its constancy shows that physicists have in mind a more trustworthy standard by which it might be judged. The ultimate standard must be available at all times and places. We require a physical structure, not necessarily permanent or transportable, but constructable at any time and place from a recorded specification.

The form of the specification is decided by the condition that the definition of length (and a corresponding definition of time interval) is required at the very beginning of physics; because the definitions of other physical quantities assume that a system of space and time measurement is already in existence. It would therefore be a vicious circle to use any 'dimensional' physical quantities in specifying the standard referred to in the definition of length. The quantitative part of the specification must consist entirely of pure numbers. The specification of physical structure by pure numbers—numbers of elementary particles in configurations or states defined by quantum numbers

[a] This subject is treated at greater length in *The Philosophy of Physical Science*, pp. 70–85.

—is developed in quantum theory. Accordingly, *the standard of length must be a quantum-specified structure.*

The equations of quantum theory determine the various spatial extensions in quantum-specified systems as fixed multiples of a unit $\hbar/m_e c$. Whether or not this unit is supposed to be constant at all times and places, the ratio of two quantum-specified extensions at the same time and place is a fixed constant. Thus all quantum-specified structures give equivalent metrics, differing from one another only by a constant conversion factor.

It remains to show that the unique quantum-specified metric is the same as the σ-metric. This follows at once if we can show that any one quantum-specified extension has a mathematically calculable, and therefore fixed, ratio to the local uncertainty σ. Since the main purpose of this book is to investigate in detail the way in which the extensions of various simple structures are related to σ, ample proof will be furnished in due course. For example, we shall find that the Rydberg constant for hydrogen $\mathfrak{R}$, which is the reciprocal of a length, is given by

$$\mathfrak{R}^{-1} = \frac{16\pi\sqrt{5}}{3}\,.\,136^2\,.\,137\,.\,\sigma. \tag{4·1}$$

Thus the use of the wave-length of the H_α line as a standard of length available at all times and places is equivalent to using the σ-metric.

For molar measurement the standard is commonly embodied in a rod, which is understood to be calibrated by means of the H_α (or some other quantum-specified) wave-length. Or we may use the extension of a fixed number of lattice spaces in a specified kind of crystal at a temperature specified in some absolute way. The standard of time is likewise defined by periods of light waves or of the vibrations of a crystal. Evidently, in replacing the Paris metre by a wave-length or crystal-lattice standard, and the earth's erratic time-keeping by a quartz clock, the practical metrologist accepts the quantum-specified standard as his ideal, so that there is no difference in the accepted meaning of length and time-interval in theoretical and experimental physics.

The ratio of the wave-length to the period of H_α light is the velocity of H_α light. Thus it follows from the definition of the ultimate standards of length and time that the velocity of light is constant everywhere and everywhen. Alleged experimental evidence for a rather large change of the velocity of light in the last 70 years has been put forward. From the nature of the case there can be no such evidence; if anything is put in doubt by the experimental results, it is the agreement of the standards used by the various observers. More baleful, because it has received more credence, is the speculation of various writers that the velocity of light has changed slowly in the long periods of cosmological time, which has seriously distracted the sane development of cosmological theory. The speculation is nonsensical because a change of the velocity of light is self-contradictory.

It is perhaps not superfluous to add that no question arises as to whether the standard here defined *really* has the same length at all times and places. The question implies that there is a more ultimate standard, invested with 'reality'—whatever that may mean—which would show up the variations, if any, of the quantum-specified standard. The concept of length must be kept free from this kind of metaphysical embroidery.

Length, like other physical quantities, is a term introduced for the purpose of succinct description of observational knowledge; and, if it is defined appropriately for this purpose, no other criticism is relevant.

5. Range of nuclear forces and the recession of the galaxies

The simplest manifestation of the uncertainty of the local physical origin occurs when we consider two particles very close together, as in a nucleus or in the close encounters of two protons in scattering experiments. If ξ_r, ξ_s are physical coordinates of the two particles, their relative position is usually described by the coordinate-difference $\xi_{rs} = \xi_s - \xi_r$. But it is also possible to measure the relative coordinate directly from one particle to the other without the intermediary of an origin. The directly measured relative coordinate will be called ξ'_{rs}. Both ξ_{rs} and ξ'_{rs} are observables, and they have the same mean value; but their probability distributions are different, that of ξ_{rs} having the greater spread. Thus the wave functions associated with them, and the conjugate momenta, are different.

An observation of ξ_r gives the distance from an undetermined point in the probability distribution of the origin to an undetermined point in the probability distribution of the particle. If ξ_s is also measured, the measure has an independent starting point in the probability distribution of the origin. Thus $\xi_s - \xi_r$ will include the coordinate-difference of two random points in the distribution of the origin; this is a quantity having a Gaussian probability distribution with standard deviation $\sigma\sqrt{2}$. By making the measurements directly from one particle to the other we eliminate this source of scatter; hence, in the notation of the theory of errors,

$$\xi_{rs} = \xi'_{rs} \pm \sigma\sqrt{2}. \tag{5·1}$$

This illustrates a principle of wide importance. The description of physical systems by probability distributions requires precautions which are liable to be overlooked because they have no counterpart in the classical conception of physics from which most of our nomenclature is derived. Definitions have to be refined to take account of distinctions unprovided for in classical terminology. This applies even to the distance between two particles, where it is necessary to state explicitly which of two quantities

$$r_{12} = (\xi_{12}^2 + \eta_{12}^2 + \zeta_{12}^2)^{\frac{1}{2}} \quad \text{and} \quad r'_{12} = (\xi'^2_{12} + \eta'^2_{12} + \zeta'^2_{12})^{\frac{1}{2}}$$

is meant. The difference is insignificant unless we are dealing with distances of the order of nuclear dimensions; but in the nucleus it is essential to distinguish r_{12} and r'_{12}. Thus, when a writer uses the term 'range of nuclear forces', we have to ask whether he means range in r_{12} or range in r'_{12}.

Normally the relative coordinates employed in quantum theory are ξ_{12}, η_{12}, ζ_{12}. In particular, the Coulomb energy is e^2/r_{12}. The non-Coulombian energy, however, is a singular energy associated with $r'_{12} = 0$, i.e. with actual coincidence of the particles. The whole electrical energy can therefore be expressed as $e^2/r_{12} + B\delta(r'_{12})$, where δ is Dirac's δ-function (§49). By (5·1) the values ξ'_{12}, η'_{12}, $\zeta'_{12} = 0$ correspond to ξ_{12}, η_{12}, $\zeta_{12} = \pm\sigma\sqrt{2}$; so that the point $r'_{12} = 0$ has a Gaussian probability distribution with standard deviation $\sigma\sqrt{2}$ over ξ_{12}, η_{12}, ζ_{12}, and $B\delta(r'_{12})$ is transformed into $Ae^{-r_{12}^2/k^2}$, where k (which is $\sqrt{2}$ times the standard deviation) is equal to 2σ. This is the form in which the non-Coulombian energy appears in the usual equations. We call k the *range constant* of nuclear forces. By (3·8),

$$k = 2\sigma = R_0/\sqrt{N}. \tag{5·2}$$

The range is simply the effect of the uncertainty of the reference frame, which scatters the singularity $r'_{12} = 0$ into a Gaussian distribution of r_{12}.

Since the range constant has been determined experimentally, chiefly from the scattering of protons by protons, and the cosmological constants R_0, N have been determined by astronomical observation of the recession of the extra-galactic nebulae, we are able, even at this early stage, to apply an observational test to the theory. The well-known formula, first derived by Einstein in 1916, for the mass M of an Einstein universe is

$$\kappa M/c^2 = \tfrac{1}{2}\pi R_0, \tag{5·3}$$

where κ is the constant of gravitation and c the velocity of light. The number of particles (protons and electrons) being N, we have $M = \frac{1}{2}N\mathrm{M}$, where M is the mass of a hydrogen atom. Hence

$$R_0/N = \kappa \mathrm{M}/\pi c^2 = 3{\cdot}95 \times 10^{-53}\,\text{cm}. \tag{5·41}$$

The experimental determination of the range constant from the scattering of protons by protons gives

$$R_0/\sqrt{N} = k = 1{\cdot}9 \times 10^{-13}\,\text{cm}. \tag{5·42}$$

From (5·41) and (5·42) we can obtain N and R_0 separately, and hence find the limiting speed of recession of the galaxies which by Lemaître's formula is $V_0 = c/R_0\sqrt{3}$. The result is $V_0 = 585$ km. per sec. per megaparsec. The actual speed should be rather less than the limiting speed, but the difference is not very important.[a] The observed value, found by Hubble and Humason, is 560 km. per sec. per megaparsec.

The observational determinations of k and V_0 do not claim high accuracy; and an agreement within 10 per cent would have been considered satisfactory. The test is therefore rather rough. But it is of particular interest because it straddles the whole range of physical systems from the nucleus to the cosmos.

Since $k = 2\sigma$, a much more accurate value of k (correct to 8 significant figures if we wish) can be obtained from (4·1). The result is $k = 1{\cdot}921 \times 10^{-13}$. This gives $V_0 = 572{\cdot}4$ km. per sec. per megaparsec.

Reversing the argument, we can deduce from the observational data that the range in r'_{12} is zero; so that non-Coulombian energy is definitely associated with a singularity of r'_{12}. Thus we need not hesitate to reject the 'meson-field' hypothesis altogether. It is in any case quite unnecessary in genuinely relativistic quantum theory. It is not an alternative way of taking into account the uncertainty of the origin, because it gives an energy distribution $Ae^{-\lambda r_{12}}$ instead of $Ae^{-r_{12}^2/k^2}$.[b]

6. Spherical space

The formula for ds in spherical space has alternative forms corresponding to different definitions of the coordinate r. The form (3·7) is obtained when we project the points

[a] From our present knowledge of the average density of matter throughout space, it is estimated that the present radius of the universe is $5R_0$. This will make the actual speed 30 km. per sec. per megaparsec less than the limiting speed V_0. (*Monthly Notices, R.A.S.* **104**, 203.)

[b] It may be expected that the shape of the non-Coulombian potential well will, at a not distant date, be determined experimentally. This will provide a crucial test between the present theory and meson-field theory.

of spherical space *orthogonally* on the tangent flat space at the origin, and take the polar coordinates in the tangent space as r, θ, ϕ. This leads to a simple graphical representation of our results.

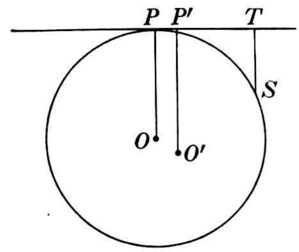
We start with a geometrical origin P and rectangular coordinates x, y, z in flat space. Let the coordinates of a particle at T be x_r, y_r, z_r. When the extraordinary fluctuation is represented by curvature, x_r, y_r, z_r are unaltered, but a fourth coordinate u_r is introduced which displaces T to a point S on the hypersphere. Transferring the origin to the centre O of the hypersphere, the equation of the hypersphere is

$$x^2+y^2+z^2+u^2 = R_0^2. \tag{6·1}$$

For a particle with uniform probability distribution over the hypersphere the mean values are

$$\overline{x_r^2} = \overline{y_r^2} = \overline{z_r^2} = \overline{u_r^2} = \tfrac{1}{4}R_0^2.$$

Thus the standard deviation of a coordinate of a particle (from its mean value 0) is $\frac{1}{2}R_0$; and the standard deviation of a coordinate of the centroid O' of the N particles is $R_0/2\sqrt{N}$, which is equal to σ by (3·8). We denote the components of OO' by x_0, y_0, z_0, u_0. Each has a Gaussian distribution with standard deviation σ.



Let P' be the orthogonal projection of O'. Since PP' is of order 10^{-13} cm., we can regard P' indifferently as a point in the tangent space or in the curved space. Its coordinates x_0, y_0, z_0 have a dispersion σ, agreeing with that of the physical origin. Thus the definition of the physical origin as the centroid of N particles is extended to particles in spherical space by simply ignoring the u coordinate. The fourth component u_0 of OO' represents the scale fluctuation of R_0. Since the radius of the hypersphere determines the linear scale of the whole universe, we naturally associate with the distinction between the geometrical origin P and the physical origin P' a distinction between the geometrical scale OP and the physical scale $O'P'$. We have

$$O'P'/OP = (R_0-u_0)/R_0.$$

Since the standard deviation of u_0/R_0 is $\sigma/R_0 = \sigma_\epsilon$, the scale fluctuation is correctly represented.

What we have here shown is that a rather naïve interpretation of the four-dimensional picture turns out to be correct. This makes four-dimensional theory rather easier than we had a right to expect. I do not think that an alternative proof of the results in § 3 can be obtained in this way. If central or stereographic projection is substituted for orthogonal projection, the standard deviation of the centroid of the projected points is not equal to $R_0/2\sqrt{N}$. It would be difficult to justify the preference for orthogonal projection without reference to the full investigation in § 3.

According to general relativity theory local irregularities of curvature are superposed on the cosmical hypersphere. It might seem that the next step in unified theory would be to derive these local curvatures from statistical fluctuations by some extension of the theory of § 3. But that would be a misunderstanding of the relation between molar and microscopic theory, and of the relation of unified theory to both. Normally the formulae of general relativity theory which covers molar physics and of quantum theory which covers microscopic physics are not comparable. Being intended for different fields of application, they introduce different kinds of approximation; so that usually when a problem is such that the approximations of general relativity theory are valid, the approximations of quantum theory are invalid, and vice versa. To unite the two theories we have to seek out special conditions in which the approximations of both are satisfied, so that the methods of both are rigorously applicable. A uniform steady distribution, or Einstein universe, provides such a meeting point.

To show how the approximations of the two theories diverge, let us consider the most typical feature of quantum theory. *Quantisation* is a complication which arises from uniformity and symmetry; for in these conditions there is a persistence of certain dynamical integrals (usually integrals of angular momentum) which invalidates the assumption on which the practice of molar averaging is based. A slight non-uniformity is treated in quantum theory as a perturbation, which does not modify the integrals though it reduces the time that they persist. Thus as the non-uniformity increases the importance of quantisation fades away. When it appears that quantal effects are no longer important, the conditions for the usual molar averaging are satisfied; and, by a discontinuous change of method, we pass over to the representation of non-uniformity by irregular curvature.

To show more definitely the incompatibility of method, consider an atom in the slightly non-uniform environment which corresponds to an irreducible gravitational field. The non-uniformity would be treated in quantum theory as a perturbation having no effect on the eigenstates of the atom but inducing transitions between them. The eigenstates are determined by a wave equation which, when expressed in tensor form, contains the tensor $g_{\mu\nu}$. Since the eigenstates are the same with or without the non-uniformity, so also are the wave equation and the coefficients $g_{\mu\nu}$ contained in it. But this directly conflicts with general relativity theory which represents the gravitational field by modifying the $g_{\mu\nu}$.

To take account of an irreducible gravitational field in the wave equation by using the $g_{\mu\nu}$ *which represent the gravitational field in molar theory would be, not a refinement, but an error.*[a] The principle of equivalence does not apply. Formally this remains true for structures so extensive that the molar $g_{\mu\nu}$ differ considerably from the uniform $g_{\mu\nu}$ used in the wave equation; but, since a wide deviation implies that transitions between the eigenstates are very frequent, the wave analysis ceases to be useful. This is the fading out of quantisation already mentioned, which leaves us free to change our method and redescribe the system in terms of the non-uniform $g_{\mu\nu}$ of molar theory.

The distinction between 'special' and 'general' relativity theory is well known. In considering the connection with quantum theory, it would be useful to distinguish also an 'intermediate' relativity theory. *Special theory* is limited to flat space-time; *intermediate theory* is an extension to curved but uniform space-time; *general theory* is a further extension to non-uniform curvature. It is intermediate theory that links up with quantum theory. Since the formulae of general relativity cover intermediate relativity, they will be used from time to time in our development of quantum theory, but always in their particular application to uniform curvature.

[a] Thus attempts to 'extend Dirac's wave equation to general relativity' are misguided, but probably the intention is only to extend it to generalised coordinates in flat space by putting it into tensor form. This is a purely mathematical transformation in no way dependent on the theory of relativity.

7. Uranoids

For the purpose of investigation we divide the universe into two parts, namely, an *object-system* and its *environment*. The term 'object-system' (object-particle, object-field, etc.) is used to distinguish the part that is being intensively studied. The environment comprises everything not specifically included in the object-system, whether surrounding it or permeating it. It might alternatively be described as the 'background'.

The environment must never be left out of consideration. It would be idle to develop formulae for the behaviour of an atom in conditions which imply that the rest of the matter of the universe has been annihilated. In relativity theory we do not recognise the concept of an atom as a thing complete in itself. We can no more contemplate an atom without a physical universe to put it in than we can contemplate a mountain without a planet to stand it on.

The most elementary formulae of physics relate to very simple object-systems in very simple environments. Just as we have to begin with very simple objects—electrons, two-particle systems, etc.—so we have to begin with very simple environments—uniform, electrically neutral, etc. These simple environments will be called *uranoids*. A uranoid is an ideally simplified universe just as a geoid is an ideally simplified earth, and it is used in an analogous way.

The uranoid adopted as standard environment for our object-systems is naturally taken to be a steady uniform probability distribution of particles. This, as we have seen, constitutes an Einstein universe, and occupies a hyperspherical space. Usually it is further specialised as a 'zero-temperature uranoid' so that the particles are at almost exact rest.[a] The advantage of zero temperature is that the environment then consists of material particles only; whereas if the temperature is not zero it includes radiation. The standard uranoid is also taken to be electrically neutral; so that if a molar electromagnetic field has to be considered, it must be included in the object-system.

The *whole universe*, usually idealised as a standard uranoid, is a partner in every problem. That does not mean that we attribute to the remote environment any greater share in determining local phenomena than is ordinarily admitted in relativity theory. In particular, the most radical change in the distribution of the extra-galactic nebulae only affects small-scale systems to the extent to which it alters $g_{\mu\nu}$ in the locality considered—an effect almost entirely eliminated by a local transformation of coordinates. We include the whole environment in order to save the trouble of dividing it. For, if we introduce a boundary, we give ourselves the extra trouble of discovering boundary conditions which shall have the same effect as a continuation of the environment beyond the boundary.

Two lines of approach have led us to consider a system of a very large number of particles in conjunction with the small system that is being intensively studied. In § 2 it was a question of *metric*; the large system determines the uncertainty of the physical reference frame, and hence the scale of the various structures in that frame. Now it is a question of *mechanics*; the environment of the object-system is actually a vast assemblage of particles, and we have to consider the physical interaction. But these two effects are really identical. Einstein's theory, by unifying geometry and mechanics,

[a] For the significance of 'almost exact' see § 10.