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V. G. Maz'ya and
T. O. Shaposhnikova

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Preface

The subject of this book is the theory of multipliers in certain classes of differentiable functions which often occur in analysis. It is intended for students and researchers who are interested in function spaces and their relation to partial differential equations and operator theory. We discuss the description of multipliers, their properties and applications. The theory of Fourier multipliers is not dealt with in this book. A knowledge of basic Sobolev space theory is assumed: we confine ourselves here to the necessary statements, but the reader who is interested in proofs can find them in monographs by, for instance, Stein [1], S. Nikol'skii [1], Peetre [1] and Triebel [3], [4].

The monograph contains seven chapters, of which the first three concern multipliers in pairs of integer and fractional Sobolev spaces, Bessel potential spaces etc. Here we present conditions for a function to belong to different classes of multipliers. Topics related to the spaces of multipliers, which are also treated in Chapters 1–3, include imbedding and composite function theorems and the spectral properties of multipliers. The applications considered include the calculus of singular integral operators whose symbols are multipliers, as well as coercive estimates for solutions of elliptic boundary value problems in spaces of multipliers.

In Chapter 4 we study the essential norm of a multiplier. The trace and extension theorems for multipliers in Sobolev spaces are proved in Chapter 5. The next chapter deals with multipliers in a domain. In particular, we discuss the change of variables in norms of Sobolev spaces and present some implicit function theorems for multipliers. Chapter 7 presents applications of these results to L_p theory of elliptic boundary value problems in domains with non-smooth boundaries.

A more detailed description of the contents is given in the Introduction.

Throughout the book there is constant cross-reference between chapters and sections. Where there is more than one theorem, lemma, proposition etc. within a subsection, these are numbered in independent sequence and in cross-reference we simply refer to the individual numbered item: for example, Theorem 3.6.2/1 is Theorem 1 in Subsection 3.6.2, Lemma 3.4.2 is the single lemma in Subsection 3.4.2 and (3.9.2/17) is formula (17) of Subsection 3.9.2.

We conclude the book with a list of symbols and author and subject indexes. The bibliography contains only papers referred to in the text and published up to 1980.

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We take pleasure in thanking Dr I. E. Verbitskii who, when the manuscript had already been sent to the publishers, kindly supplied us with the proof—just obtained—of our hypothesis on a simplified normalization of the space $M(H_p^m \to H_p^l)$ for $m > l$. This material forms Section 2.6.

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Leningrad
September 1984

Vladimir Maz'ya
Tatyana Shaposhnikova

Introduction

By a multiplier acting from one functional space, S_1, into another, S_2, we mean a function which defines a bounded linear mapping of S_1 into S_2 by pointwise multiplication. Thus, with any pair of spaces S_1, S_2 we associate a third—the space of multipliers $M(S_1 \to S_2)$.

Multipliers arise in various problems of analysis and theory of differential and integral equations. Coefficients of differential operators can be naturally considered as multipliers. The same is true for symbols of more general pseudo-differential operators. Multipliers also appear in the theory of differentiable mappings preserving the Sobolev spaces. Solutions of boundary value problems can be sought in classes of multipliers. Because of their algebraic properties, multipliers are suitable objects for a generalization of the basic facts of calculus (theorems on superposition, implicit function theorems etc.).

The present book is devoted to the theory of multipliers in pairs of integer and fractional Sobolev spaces, Bessel potential spaces etc. Regardless of the substantiality and the usefulness of this theory, until recently it has attracted relatively little attention. The earliest papers devoted to the subject include that due to Devinatz and Hirschman [1], 1959, on the spectrum of the operator of multiplication in the space W_2^l, $2|l|<1$, on the unit circumference; two papers by Hirschman, [2], 1961 and [3], 1962, which also treat multipliers in W_2^l; and, finally, a study of multipliers in Bessel potential space due to Strichartz [1], 1967.

This monograph is based principally on the authors' results obtained in 1979–1980 and partly published in journals. The outline of the basic results given below is not exhaustive and does not follow the strict sequence of the subsequent exposition.

Description and Properties of Multipliers

According to a theorem due to one of the authors (Maz'ya [1], 1962), the inequality

$$\int_\Omega |\gamma(x)u(x)|^2 \, dx \leq \int_\Omega |\nabla u(x)|^2 \, dx \tag{1}$$

is valid for all $u \in C_0^\infty(\Omega)$ if, for any compact set e with $e \subset \Omega \subset R^n$,

$$\int_e |\gamma(x)|^2 \, dx \leq \frac{n-2}{4} \omega_n \operatorname{cap}_\Omega(e).$$

Moreover, the inequality

$$\int_e |\gamma(x)|^2 \, dx \leq (n-2)\omega_n \operatorname{cap}_\Omega(e)$$

is necessary for the validity of (1). Here $n>2$, ω_n is the surface area of a unit ball in R^n and $\operatorname{cap}_\Omega(e)$ is the Green capacity of the compact set e relative to Ω. Thus the condition

$$\sup_{e \subset \Omega} \frac{\|\gamma; e\|_{L_2}}{[\operatorname{cap}_\Omega(e)]^{1/2}} < \infty$$

is equivalent to the inclusion $\gamma \in M(\mathring{L}_2^1(\Omega) \to L_2(\Omega))$ where $\mathring{L}_2^1(\Omega)$ is the completion of C_0^∞ in the metrics of the Dirichlet integral.

This result serves as a model for the characterization of different spaces of multipliers which are presented in the book. For example, multipliers $\gamma: W_p^m(R^n) \to W_p^l(R^n)$ satisfy the relation

$$\|\gamma; R^n\|_{M(W_p^m \to W_p^l)} \sim \sup_{e \subset R^n} \frac{\|\nabla_l \gamma; e\|_{L_p}}{[\operatorname{cap}(e, W_p^m)]^{1/p}} + \sup_{e \subset R^n} \frac{\|\gamma; e\|_{L_p}}{[\operatorname{cap}(e, W_p^{m-l})]^{1/p}} \qquad (2)$$

where $p \in (1, \infty)$, $m \geq l \geq 0$, ∇_l is the gradient of order l and $\operatorname{cap}(e, W_p^k)$ is the capacity generated by the norm in $W_p^k(R^n)$. For the case $m=l$ this implies the equivalence

$$\|\gamma; R^n\|_{MW_p^l} \sim \sup_{e \subset R^n} \frac{\|\nabla_l \gamma; e\|_{L_p}}{[\operatorname{cap}(e, W_p^l)]^{1/p}} + \|\gamma; R^n\|_{L_\infty}. \qquad (3)$$

(Here and in what follows, $MS = M(S \to S)$.) The finiteness of the right-hand side of (2) is the necessary and sufficient condition for a function γ to belong to the space $M(W_p^m(R^n) \to W_p^l(R^n))$.

Recently I. E. Verbitskii showed that, for $m>l$, the second summand on the right in (2) can be replaced by $\sup_x \|\gamma; B_1(x)\|_{L_1}$. His result is presented in the last section of Chapter 2 within the more general context of Bessel potential spaces.

In two cases, $p=1$ and $pm>n$, we can describe the space $M(W_p^m(R^n) \to W_p^l(R^n))$ using no capacity. For $p=1$, the ball $B_\rho(x) = \{y \in R^n : |x-y| < \rho\}$, $\rho < 1$, plays the role of an arbitrary compact set e in (2). In other words,

$$\|\gamma; R^n\|_{M(W_1^m \to W_1^l)} \sim \sup_{x \in R^n, 0<\rho<1} (\rho^{m-n} \|\nabla_l \gamma; B_\rho(x)\|_{L_1} + \rho^{m-l-n} \|\gamma; B_\rho(x)\|_{L_1}). \qquad (4)$$

In the case $pm > n$, one can change e by $B_1(x)$ in (2), i.e.

$$\|\gamma; R^n\|_{M(W_p^m \to W_p^l)} \sim \|\gamma; R^n\|_{W^l_{p,\mathrm{unif}}}. \tag{5}$$

Here and henceforth

$$\|\gamma; R^n\|_{S_{\mathrm{unif}}} = \sup_{z \in R^n} \|\gamma \eta_z; R^n\|_S,$$

where $\eta_z(x) = \eta(z-x)$, η is an arbitrary function in $C_0^\infty(R^n)$, $\eta = 1$ on the ball $B_1 = \{x : |x| < 1\}$.

The greater part of Chapter 1 is devoted to the proof of the above-stated assertions.

Similar results for the pair $H_p^m(R^n) \to H_p^l(R^n)$ of Bessel potential spaces and the pair $W_p^m(R^n) \to W_p^l(R^n)$ of fractional order Sobolev spaces are obtained in the following two chapters. Relations analogous to (2)–(4), in which the corresponding 'fractional derivative' plays the role of the gradient, are valid for the spaces $M(H_p^m(R^n) \to H_p^l(R^n))$, $p \in (1, \infty)$, and $M(W_p^m(R^n) \to W_p^l(R^n))$, $p \in [1, \infty)$. The relation (5) can also be extended to the above-mentioned spaces of multipliers which, in particular, contains the result due to Strichartz [1], 1967: $MH_p^l(R^n) = H^l_{p,\mathrm{unif}}(R^n)$ for $pl > n$.

Relations similar to (2), together with upper and lower estimates for capacity, enable one to obtain necessary or sufficient conditions for being a multiplier which are formulated without the use of capacity. For example, in Chapter 2 we show that the space $M(H_p^m(R^n) \to H_p^l(R^n))$ with $m \geq l$, $mp \leq n$, contains functions which are 'uniformly locally' in the space $B_{q,\infty}^{n/q-m+l}$, where $q \geq p$, $\{n/q - m + l\} > 0$, $n/q > m$. In the case $m = l$ we additionally require the boundedness of functions (for the space $B^s_{p,r}$ see, for instance, Nikol'skii [1], Peetre [1]). In particular, for the one-dimensional case the last result implies a sufficient condition in terms of q-variation which generalizes one due to Hirschman [2], 1962.

In Chapter 2 we also show that $H^l_{n/m,\mathrm{unif}}(R^n) \subset M(H_p^m(R^n) \to H_p^l(R^n))$ for $mp < n$, $l < m$ and $(H^l_{n/l,\mathrm{unif}} \cap L_\infty)(R^n) \subset MH_p^l(R^n)$ for $lp < n$. The latter imbedding was proved earlier, by a direct method, by Polking [1]. In Chapter 3 we prove that H can be replaced by W in the right-hand side only for $p \geq 2$.

In different parts of the book we study certain properties of multiplier spaces. We prove imbedding theorems of the form $M(H_p^m(R^n) \to H_p^l(R^n)) \subset M(H_q^{m-j}(R^n) \to H_q^{l-j}(R^n))$, theorems on the composition $\varphi(\gamma)$ where γ is a multiplier, on the spectrum of a multiplier etc. For instance, in 2.5 a description of the point, residual and continuous spectra of multipliers in H_p^l and $H_{p'}^{-l}$ is presented. In Chapter 3 the following assertion, generalizing the theorem of Hirschman [1], is given: if a function φ satisfies the Hölder condition, $|\varphi(t+\tau) - \varphi(t)| \leq A\,|\tau|^\rho$, $|\tau| < 1$, with $\rho \in (0, 1)$ and $\gamma \in M(W_p^m(R^n) \to W_p^l(R^n))$, $m \geq l$, $0 < l < 1$, $p > 1$, then $\varphi(\gamma) \in M(W_p^{m-l+r}(R^n) \to W_p^r(R^n))$ for any $r \in (0, l\rho)$.

In Chapter 6 we obtain equivalent expressions for the norm in the space $M(W_p^m(\Omega) \to W_p^l(\Omega))$, where m and l are integers, $m \geq l \geq 0$ and Ω is a Lipschitz domain. We prove that for such domains there exists a linear continuous extension operator $M(W_p^m(\Omega) \to W_p^l(\Omega)) \to M(W_p^m(R^n) \to W_p^l(R^n))$, $p \geq 1$, and that $M(W_p^m(\Omega) \to W_p^l(\Omega))$ is the space of restrictions to Ω of functions in $M(W_p^m(R^n) \to W_p^l(R^n))$. For an arbitrary domain the last property is not valid.

In Chapter 6 we also define and study the so-called (p, l)-diffeomorphisms, i.e. the bi-Lipschitz mappings $\varkappa : R^n \supset U \to V \subset R^n$ such that the elements of the Jacobi matrix $\varkappa'$ belong to the space of multipliers $MW_p^{l-1}(U)$. These mappings have a number of useful properties. They map $W_p^l(V)$ onto $W_p^l(U)$ and $\varkappa^{-1}$ is the (p, l)-diffeomorphism together with $\varkappa$. Moreover, the class of (p, l)-diffeomorphisms is closed with respect to a composition of mappings.

Using these properties of (p, l)-diffeomorphisms, we introduce the class of n-dimensional (p, l)-manifolds, on which the space W_p^l is properly defined.

One more subject considered in Chapter 6 is the class $T_p^{m,l}$ of mappings $\varkappa : U \to V$ which satisfy the inequality $\|u \circ \varkappa; U\|_{W_p^l} \leq c\,\|u; V\|_{W_p^m}$.

One of implicit function theorems proved in Chapter 6 is the following.

Let $G = \{(x, y) : x \in R^{n-1}, y > \varphi(x)\}$ *and let u be a function on G such that*
(i) grad $u \in MW_p^{l-1}(G)$, *where l is an integer*, $l \geq 2$,
(ii) $u(x, \varphi(x)) = 0$,
(iii) $\inf\, (\partial u/\partial y)(x, \varphi(x)) > 0$.
Then grad $\varphi \in MW_p^{l-1-1/p}(R^{n-1})$.

Essential Norm of a Multiplier

The norm in the quotient space of the space of multipliers modulo compact operators will be called the essential norm of a multiplier. In Chapter 4 we give two-sided estimates for the essential norm

$$\text{ess}\,\|\gamma; R^n\|_{M(W_p^m \to W_p^l)} = \inf_{\{T\}} \|\gamma - T\|_{W_p^m \to W_p^l},$$

where $\{T\}$ is the set of all compact operators: $W_p^m \to W_p^l$, $p \geq 1$, and m, l are simultaneously integer or fractional numbers. For example, if $p > 1$,

$mp \leq n$, m and l are integers, then

$$\text{ess}\,\|\gamma; R^n\|_{M(W_p^m \to W_p^l)} \sim \lim_{\delta\to 0} \sup_{\{e:\text{diam}\, e\leq\delta\}} \left(\frac{\|\nabla_l\gamma; e\|_{L_p}}{[\text{cap}\,(e, W_p^m)]^{1/p}} + \frac{\|\gamma; e\|_{L_p}}{[\text{cap}\,(e, W_p^{m-l})]^{1/p}}\right),$$

$$+\lim_{r\to\infty} \sup_{\{e\subset R^n\setminus B_r:\,\text{diam}\, e\leq 1\}} \left(\frac{\|\nabla_l\gamma; e\|}{[\text{cap}\,(e, W_p^m)]^{1/p}} + \frac{\|\gamma; e\|_L}{[\text{cap}\,(e, W_p^{m-l})]^{1/p}}\right). \tag{6}$$

For $mp > n$ the right-hand side of this relation has a simpler form, namely,

$$\text{ess}\,\|\gamma; R^n\|_{M(W_p^m\to W_p^l)} \sim \overline{\lim_{|x|\to\infty}}\, \|\gamma; B_1(x)\|_{W_p^l} \quad \text{for } m > l,$$

$$\text{ess}\,\|\gamma; R^n\|_{MW_p^l} \sim \|\gamma; R^n\|_{L_\infty} + \overline{\lim_{|x|\to\infty}}\, \|\gamma; B_1(x)\|_{W_p^l} \quad \text{for } m = l.$$

Making the right-hand sides of these and similar relations equal to zero, we obtain the characteristics of the space $\mathring{M}(W_p^m(R^n) \to W_p^l(R^n))$, $m > l$, of compact multipliers. In 4.2 we show that this space coincides with the completion of $C_0^\infty(R^n)$ in the norm of the space $M(W_p^m(R^n) \to W_p^l(R^n))$. In accordance with the latter assertion, by $\mathring{M}W_p^l(R^n)$ we denote the completion of $C_0^\infty(R^n)$ with respect to the norm of the space $MW_p^l(R^n)$. By virtue of a theorem proved in 4.3, the essential norm in $\mathring{M}W_p^l(R^n)$ is equivalent to the norm in $L_\infty(R^n)$. In addition, we note that for any multiplier in $W_p^l(R^n)$ the inequality

$$\|\gamma; R^n\|_{L_\infty} \leq \text{ess}\,\|\gamma; R^n\|_{MW_p^l}$$

is valid (see 4.3).

Traces and Extensions of Multipliers

It is well known that the space $W_p^{l-1/p}(R^n)$with integer l is the space of traces on R^n of functions contained in the Sobolev space $W_p^l(R_+^{n+1})$, where $R_+^{n+1} = \{(x, y) : x \in R^n, y > 0\}$. In Chapter 5 we show that, similarly, for the space of multipliers $MW_p^{l-1/p}(R^n)$, the traces on R^n of functions in $MW_p^l(R_+^{n+1})$ belong to the space $MW_p^{l-1/p}(R^n)$ and there exists a linear continuous extension operator $MW_p^{l-1/p}(R^n) \to MW_p^l(R_+^{n+1})$. This result is contained in a more general assertion relating weighted Sobolev spaces. In the same Chapter 5 we show that, for $\{l - m/p\} > 0$, $1 \leq p < \infty$, the space $MW_p^{l-m/p}(R^n)$ coincides with the space of traces on R^n of functions in $MW_p^l(R^{n+m})$.

Applications of the Theory of Multipliers

In the course of the book we often dwell upon various applications of multipliers to the theory of differential and integral operators. In terms of multipliers we establish the bounds (in some cases two-sided) for a norm and for an essential norm of a differential operator acting in a pair of Sobolev spaces. The properties of a function in $M(W_2^m(R^n) \to W_2^l(R^n))$ are equivalent to the properties of the integral convolution operator considered as a mapping of $L_2(R^n; (1+|x|^2)^{m/2})$ into $L_2(R^n; (1+|x|^2)^{l/2})$. The basic tenets of the theory of singular integral operators acting in $W_p^l(R^n)$ can be generalized to operators with symbols belonging to spaces of multipliers. One can solve elliptic boundary value problems in spaces of multipliers. For multiplier norms of solutions, estimates similar to the known coercive L_p-estimates are valid. Here is the simplest example considered in 6.6: the Dirichlet problem

$$\Delta u = 0 \quad \text{in} \quad \Omega, \qquad u|_{\partial\Omega} = \varphi,$$

where Ω is a bounded Lipschitz domain in R^n, is uniquely solvable in the space $MW_2^1(\Omega)$ for any $\varphi \in MW_2^{1/2}(\partial\Omega)$. The solution satisfies the inequality

$$\|u; \Omega\|_{MW_2^1} \leq c\, \|\varphi; \partial\Omega\|_{MW_2^{1/2}}.$$

In the last chapter, multipliers are applied to the study of conditions on $\partial\Omega$ ensuring $W_p^l(\Omega)$-coercivity of the operator of the general elliptic boundary value problem. For $p(l-1) < n$, the condition we have found is the following: for any point of the boundary there exists a neighbourhood U and a function φ such that $U \cap \Omega = \{(x, y) \in U : x \in R^{n-1}, y > \varphi(x)\}$ and

$$\|\nabla\varphi; R^{n-1}\|_{MW_p^{l-1-1/p}} \leq \delta, \tag{7}$$

where δ is a small constant. We consider equivalent formulations of this requirement and obtain different sufficient conditions for its validity. For example, Ω satisfies (7) provided that φ has a small Lipschitz constant and belongs to the space $B_{q,p}^{l-1/p}(R^{n-1})$ for some q. In case $p(l-1) > n$, condition (7) should be replaced by $\varphi \in W_p^{l-1/p}(R^{n-1})$.

Special attention is paid to the Dirichlet boundary value problem. In particular, we show here that, for $p(l-1) > n$, the condition $\varphi \in W_p^{l-1/p}(R^{n-1})$ is not only sufficient but also necessary for solvability in $W_p^l(\Omega)$ of this problem in one of the two formulations we consider.

This is a summary of the book. The theory of multipliers in spaces of differentiable functions is still at the very beginning of its development and, undoubtedly, the discussion presented here by no means exhausts the possible scope of the theory. It suffices to remark that, simply by varying the pairs of function spaces S_1 and S_2, often encountered in analysis, we immediately arrive at new unsolved problems in the description and properties of $M(S_1 \to S_2)$.

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