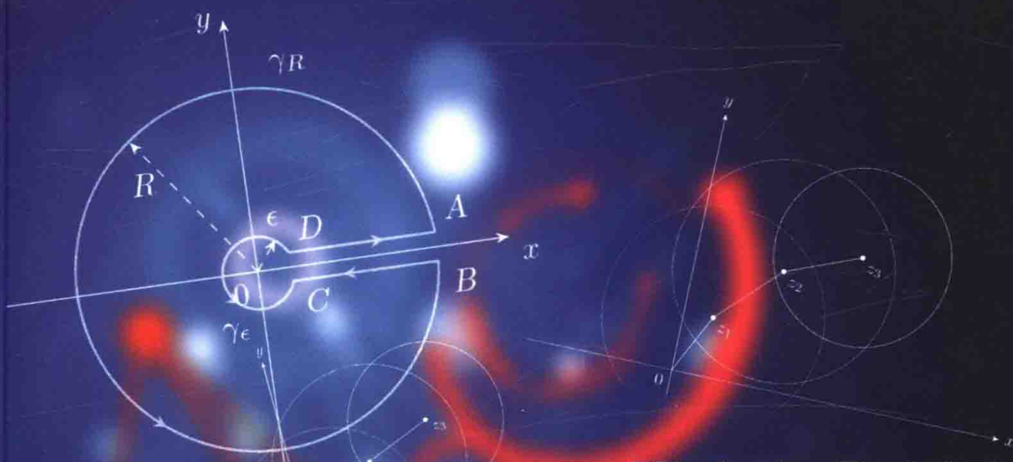
Essential Textbooks in Physics



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A Guide to Mathematical Methods for Physicists

With Problems and Solutions

World Scientific

A Guide to

Mathematical Methods for Physicists

With Problems and Solutions

Mathematics plays a fundamental role in the formulation of physical theories. This textbook provides a self-contained and rigorous presentation of the main mathematical tools needed in many fields of Physics, both classical and quantum. It covers topics treated in mathematics courses for final-year undergraduate and graduate physics programmes, including complex functions, distributions, Fourier analysis, linear operators, Hilbert spaces and eigenvalue problems. The different topics are organised into two main parts — complex analysis and vector spaces — in order to stress how seemingly different mathematical tools, for instance the Fourier transform, eigenvalue problems or special functions, are all deeply interconnected. Also contained within each chapter are fully worked examples, problems and detailed solutions.

A companion volume covering more advanced topics that enlarge and deepen those treated here is also available.

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Preface

Mathematics is a fundamental ingredient of any physical theory. This book is an introduction to the mathematics behind concrete physical problems, both in classical and quantum physics. It covers topics treated in mathematics courses in the last years of undergraduate studies in physics: complex functions, distributions and Fourier analysis, which are widely used in all fields of physics, and an introduction to the theory of Hilbert spaces, which is needed for the mathematical foundation of quantum mechanics. The different arguments are organised into two main parts — complex analysis and vector spaces. This division is meant to stress how seemingly different mathematical tools used in physics are conceptually related and deeply interconnected.

In our experience some of these topics are covered in physics programmes either at an extremely elementary level or at a very mathematically advanced one. This book originates from lectures given at the Ecole Normale Superiéure and the Université Pierre et Marie Curie in Paris, the University of Milano-Bicocca and the University of Rome Tor Vergata, and takes an intermediate approach. It keeps a rigorous mathematical level, but it uses examples to illustrate the general theory rather than showing long and complicated proofs. For the topics that involve advanced functional analysis, like the theory of linear operators in Hilbert spaces, we have emphasised the physicist point of view in spite of generality, with an eye to quantum mechanics.

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Newtonian Mechanics for Undergraduates
by Vijay Tymms

Introduction to General Relativity and Cosmology
by Christian G. Böhmer

A Guide to Mathematical Methods for Physicists: With Problems and Solutions
by Michela Petrini, Gianfranco Pradisi and Alberto Zaffaroni

Each chapter contains fully worked-out exercises. Together with the examples, they are a central part of the book and illustrate the general theory. There are about 150 exercises with solutions. They range from very simple and basic exercises, which the readers are invited to solve by themselves, to more difficult or theoretical ones, denoted with a star. The latter sometimes provide proofs of theorems given in the main text.

This book is appropriate for one semester courses for second or third year undergraduate physics programmes. It assumes the knowledge of elementary calculus in one or more real variables and some previous exposure to elementary algebra, at the level of vectors and matrices, usually covered in the first years of undergraduate studies.

A companion volume is also available, which covers more advanced topics that are typically taught at Master level. It includes conformal mapping, asymptotic analysis, integral and differential equations and an introduction to the mathematical methods of quantum mechanics. It is impossible to cover in a single book all mathematical methods used in physics with a satisfactory level of analysis. For this reason, some fundamental topics, like probability, group theory and differential geometry, are missing both in this book and its companion.

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PART I

Complex Analysis

1
Holomorphic Functions

The theory of complex functions is one of the most fascinating fields of analysis and is also an essential tool for theoretical physics. The idea is to extend to the functions of complex variable properties and theorems valid for real analysis. In particular, the aim is to define the notion of derivability and integrability of a function of complex variables. This will lead to the definition of holomorphic functions. We will discover that differentiability in complex sense is stronger than in real sense. In particular, functions that are differentiable once in complex sense are also infinitely differentiable and analytic.

1.1. Complex Functions

A function of complex variable is a map $f : \mathbb{C} \to \mathbb{C}$ that associates with each point z of the complex plane $\mathbb{C}$ (or a subset of it) one point of the same plane

$$z \to w = f(z)\,. \tag{1.1}$$

Since $f(z)$ is a complex number, we introduce its *real* and *imaginary parts* as follows:

$$f(z) = \mathrm{Re}\, f + i\, \mathrm{Im}\, f = X(x, y) + iY(x, y), \tag{1.2}$$

where $z = x + iy$, and $X(x, y)$ and $Y(x, y)$ are real functions of the two real variables (x, y).

A first notion one can introduce is continuity. A function $f(z)$ is *continuous* at a point z_0 if it is defined in a neighbourhood of z_0 and if

$$\lim_{z \to z_0} f(z) = f(z_0). \tag{1.3}$$

The limit is defined in analogy with the case of a function of one real variable: $f(z_0)$ is the limit of $f(z)$ for $z \to z_0$ if, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|z - z_0| < \delta \quad \Rightarrow \quad |f(z) - f(z_0)| < \epsilon. \tag{1.4}$$

Note however that we are taking the limit in a plane. Thus, as for functions of several real variables, the limit must be independent of the path chosen to approach z_0.

Example 1.1. The limit

$$\lim_{z \to 0} \frac{\bar{z}}{z} \tag{1.5}$$

does not exist, since it depends on the path in the complex plane chosen to approach $z = 0$. For instance, if we approach $z = 0$ along the real axis, $z = \bar{z} = x$, while along the imaginary axis $z = -\bar{z} = iy$. Thus, the limits take different values

$$\lim_{x \to 0} \frac{\bar{z}}{z} = \frac{x}{x} = 1, \quad \lim_{y \to 0} \frac{\bar{z}}{z} = \frac{-iy}{iy} = -1. \tag{1.6}$$

From (1.2), it follows that, if a function $f(z)$ is continuous at $z_0 = x_0 + iy_0$, its real and imaginary parts are also continuous at the point (x_0, y_0) of $\mathbb{R}^2$. Similarly, one can show that the modulus of f, $|f| = \sqrt{X^2 + Y^2}$, is also continuous at $z = z_0$. Moreover, if the functions $f(z)$ and $g(z)$ are continuous at $z = z_0$, and h is continuous in $g(z_0)$, then

$$f(z) \pm g(z), \quad f(z)g(z), \quad \frac{f(z)}{g(z)} \quad (\text{if}\, g(z_0) \neq 0), \quad h(g(z)) \tag{1.7}$$

are also continuous at $z = z_0$.

Given the notion of continuity at a point, one can extend it to continuity on a region of the complex plane. Let us first define a *domain* of the complex plane as an open, connected region of $\mathbb{C}$. A function $f(z)$ is said to be continuous on a domain $D \subseteq \mathbb{C}$ if it is continuous at each point of the domain. All the properties mentioned above hold on D.

Given a function $f : D \to \mathbb{C}$ that associates to any $z \in D$ a value $w = f(z)$, one can define its *inverse* as

$$f^{-1} : f(D) \to D \subset \mathbb{C}$$
$$w \to z = f^{-1}(w) \ .$$

Strictly speaking, only injective functions admit an inverse. In complex analysis it is sometimes useful to define the inverse also for non-injective functions, but, in this case, $f^{-1}(w)$ is not a function in the proper sense, since it associates multiple values of z to a single value w. With an abuse of language, we call them *multi-valued functions*. We will see many examples of multi-valued functions in Sections 1.3.2 and 1.5.

1.2. Holomorphic Functions

Another important notion we want to introduce is differentiability. We will see that, on the complex plane, differentiability imposes strong constraints on the functions and that, as a consequence, differentiable functions have very interesting properties.

1.2.1. *Derivative of a complex function*

The derivative of a complex function is defined in the same way as the derivative of a real one. Consider a complex function f that is continuous at the point z_0. The function $f(z)$ is *differentiable* at the point z_0 if the limit

$$\left.\frac{df}{dz}\right|_{z_0} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{1.8}$$

exists. We call this limit *derivative* of f in z_0. We will also use the notation $f'(z_0)$. From the definition of limit, it follows that the result must be independent of the path in the complex plane along which z goes to z_0. It is important to stress that a differentiable function is also continuous, but that the opposite is not always true.

A function $f(z)$ is differentiable on a domain D if it is differentiable at all points of D. A function that is differentiable on a domain D is said to be