DISPERSION RELATIONS

EDITED BY

G. R. SCREATON

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Scottish Universities' Summer School
1960

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G. R. SCREATON
M.A., PH.D.

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PREFACE

THE Theoretical Physics Summer Schools on the Continent of Europe have for some years past occupied a popular and important place in the calendar of scientific events. Here in Britain the wish to be able to offer something similar has frequently been expressed, and physicists in Scotland have very naturally linked this wish with a belief that a Summer School in Scotland would be particularly welcome.

The opportunity to make concrete plans came when funds to organise Advanced Study Institutes were offered by the NATO authorities. In the summer of 1959 representatives of the Physics Departments of the four Scottish Universities met informally and decided to apply for NATO funds and to begin planning a Summer School. With the approval of the authorities of all four Universities a formal organisation, The Scottish Universities' Summer School in Physics, was created, the governing committee of which consists of the Physics Professors of the Scottish Universities. Under its auspices an Executive Committee for 1960 was appointed with instructions to organise a first Summer School in the Quantum Theory of Fields with special emphasis on Dispersion Relations. The undersigned was named Director of the 1960 School and the following officers were appointed: Dr D. J. Candlin (Edinburgh), Secretary; Dr W. K. Burton (Glasgow), Treasurer; and Dr G. R. Screaton (Edinburgh), Editor.

The first problem, to find a suitable location for the school, was soon happily solved. At a very convenient distance from Edinburgh the mansion of Newbattle Abbey, formerly the home of the Lothian family, stands on the site of a mediæval monastery, which in 1936 was donated in trust by the Marquess of Lothian for the establishment of a residential Adult Education College. This mansion with its facilities is available to outside bodies during a period in the summer. The Committee felt that Newbattle Abbey promised to satisfy admirably the requirements of the Summer School and without hesitation decided to hold the School there.

The selection of speakers and the detailed planning of the syllabus presented some difficulties, mainly because definite planning inevitably started rather late, but also because it was discovered that our provisional programme overlapped quite considerably with that of the Les Houches Summer School. Ultimately, however, this latter circumstance proved to be an advantage rather than the reverse in that it was possible for lecturers to contribute to the two Schools in succession, thanks to most friendly cooperation from the organisers of the Les Houches School.

The likely demand for student places at our School was at the outset a matter of guesswork, but as it turned out applications far exceeded the numbers that could be accepted. One consequence of this was that all the selected students were highly expert, so that the level of the courses required became very advanced. Thus almost automatically, the theme of the School came to be narrowed down to Dispersion Relations. At this advanced level three weeks were barely sufficient to cover the subject adequately.

The generous, and essential, financial support from NATO made it possible not only to attract a most authoritative and suitable group of lecturers, but also to reduce the price charged to students for residence at Newbattle and award a certain number of Bursaries. The funds also proved sufficient to make our care for the participants go somewhat beyond the provision of bare necessities, so that, apart from the larger excursions for which a charge was made, a considerable variety of entertainments could be laid on (not least among them the introduction of our guests to Scottish Country Dancing and to the mysteries of the Bagpipes).

To the scientific value of the School this volume bears testimony. Encouraged by what is believed to be the success of the first venture, the Scottish University Summer School for 1961 is now being prepared. It is to be on the subject of "Noise, Relaxation and Resonant Absorption (particularly in Magnetic Systems)" and its Director is Dr G. A. Wyllie of Glasgow. It will again be held at Newbattle Abbey.

It remains for me, speaking on behalf of all concerned, to thank the many persons who made the 1960 Summer School what it was:

The NATO authorities who made the School possible;

The team of lecturers, Drs Chew, Frazer, Fubini, Jackson, Jauch, Moravcsik, Polkinghorne and Thirring, to which names should be added that of Dr M. L. Goldberger who, though prevented by illness in his family from lecturing to the School, gave much help and advice at the planning stage;

The officers of the School (named above) and the other members of the Executive Committee; Prof. Gunn, Dr Higgs and Dr Strachan who helped actively in the running of the School. Particular thanks are due to Dr D. J. Candlin who bore the brunt of organising work so willingly and efficiently;

The postgraduate students of the Tait Institute of Mathematical Physics, Edinburgh, who helped in the organisation in a great many capacities, but especially, together with Mrs R. W. Chester, secretary, did great work in the major task of providing lecture notes punctually for all courses;

The Warden, Bursar and Staff of Newbattle Abbey College, who looked after us so well and willingly;

Lastly, all participants in the School who contributed to give the School the happiest of atmospheres.

To all these our gratitude is extended; at the same time, we express sincere regret to all those deserving applicants who could not be accepted for the School this time. We hope to make amends on future occasions.

N. Kemmer
Tait Professor of Natural Philosophy
University of Edinburgh

EDITOR'S NOTE

With the exception of the last lecture of Professor Fubini, the lecture notes were prepared by the lecturers themselves. In preparing the notes for publication a number of changes have been made and, as speedy publication was felt to be of primary importance, the lecturers were not given the opportunity to see the proofs. Consequently the Editor and his staff should be held responsible for any errors and misprints.

Professor Chew, as well as giving a lecture course at this Summer School, gave one at the Summer School of Theoretical Physics, Les Houches (July 1960). His lecture notes combine the material presented at both. In particular the sections on the electromagnetic structure of the pion and the nucleon were not presented at the Scottish Summer School, this subject being dealt with by Professor Frazer.

Executive Committee of the 1960 Summer School:

Professor N. Kemmer, F.R.S., Edinburgh, *Chairman.*
Dr W. K. Burton, Glasgow, *Treasurer.*
Dr D. J. Candlin, Edinburgh, *Secretary.*
Professor J. C. Gunn, Glasgow.
Dr P. W. Higgs, Edinburgh.
Dr G. R. Screaton, Edinburgh, *Editor.*
Dr C. Strachan, Aberdeen.

Lecturers:

Dr G. F. Chew, Berkeley.
Dr W. R. Frazer, Berkeley.
Dr S. Fubini, Padua.
Dr J. C. Jackson, Illinois.
Professor J. M. Jauch, Geneva.
Dr M. J. Moravcsik, Livermore.
Dr J. C. Polkinghorne, Cambridge.
Professor W. Thirring, Vienna.

Participants

Dr W. Alles, Bologna
Mr C. P. Arrhen, Lund
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Dr L. Tewordt, Munster
Mr M. J. G. Veltman, Utrecht
Mr A. E. A. Warburton, Cambridge
Mr W. J. Woolcock, Cambridge

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HISTORICAL SURVEY OF DISPERSION RELATION TECHNIQUES

TODAY the words " dispersion relations " or " spectral representations " cover a multitude of sins. But the basic thread running through all of the various manifestations is the idea that quantum mechanical amplitudes for physical processes are the boundary values of analytic functions of one or more complex variables. Right from the very beginning in the original work of Kramers, complex variable theory has played a role. And today the emphasis is more and more on the exploitation of the analytic properties of causal scattering amplitudes, combined with unitarity, as a means to calculate physically useful quantities.

The history of dispersion relations can perhaps be divided into three time intervals:

1926-53—from the original work of Kramers and Kronig to the use of their relations in quantum electrodynamics and the establishment of certain properties of the S-matrix in non-relativistic theory.

1954-58—from the first " proofs " of ordinary dispersion relations in relativistic quantum field theory, through numerous applications to theory and experiment in high energy physics, rigorous proofs, etc.

1959-*present*—from the conjecture of Mandelstam on the analytic properties of amplitudes as a function of both energy and momentum transfer, through the applications of the double dispersion relations to calculations, the investigations of the analytic properties of amplitudes in perturbation theory, and on into the future.

We will present a brief historical review of these eras, with limited references to key papers. The more recent aspects will be touched only very lightly since (*a*) there have been literally hundreds of papers written on the subject in the last few years, and (*b*) the other lecturers will give extensive references and detailed discussions of all these recent developments.

1. Early Chronology

Although there may have been earlier related considerations, the real starting point of dispersion relations occurred in 1926-27 with the historic papers of Kronig [1] and Kramers.[2] Both Kronig and Kramers were concerned with the classical dispersion of light and the relation between the real and imaginary parts of the index of refraction. Kramers' paper is the more detailed of the two; he emphasised that a specific relation between the real (dispersive) and imaginary (absorptive) parts of the index of refraction (a dispersion relation; see Section 2 below) was based on the fundamental requirement of causality. He first showed that the dispersion relation followed from the requirement that the index of refraction, viewed as a function of complex frequency, be an analytic function in the upper half plane and approach unity at infinity. Then he showed that in a medium described by such an index of refraction signals could not propagate faster than the velocity of light. There is an intimate connection between these ideas and the properties of the response function of an electrical circuit, well known to electrical engineers and first discussed by Foster, Campbell and Zobel (see *Bell System Technical Journal*, 1922-24).

Apart from an occasional application of the Kramers-Kronig relation to optical or X-ray problems (see, for example, A. H. Compton and S. K. Allison [3]) interest in the subject waned during the next twenty years.

In 1946 Kronig [4] suggested that the causality requirement might be imposed on the *S*-matrix for elementary particle processes to limit its form. Subsequently a number of investigations were made in the non-relativistic and relativistic theory in order to delimit the form of the *S*-matrix. Some of these were based on the ideas of causality.[5] Others, notably Wigner,[6] considered the properties of *S* stemming from the customary assumptions of local, Hermitean interactions, unitarity and completeness.

At about the same time (1951-52) Toll [7] and Wheller were applying the Kramers-Kronig dispersion relation to quantum electrodynamics. Such problems as the forward scattering of light by the nuclear Coulomb field (dispersive process) were related to pair production by photons in the same Coulomb field (absorptive process) in an elegant manner. Published calculations along these lines were given by Rohrlich and Gluckstern.[8]

2. The Modern Era (just past)

The next era of dispersion relations began in 1954 with the paper by Gell-Mann, Goldberger and Thirring.[9] In this paper the quantum mechanical proof of the dispersion relations for forward scattering of light was based on the *microscopic causality condition* that the commutator (anticommutator) of two boson (fermion) Heisenberg field operators at different points in space-time must vanish for space-like separations of the two points. Applications were made to the nuclear photo-effect and to the Compton scattering of gamma rays by nucleons. The proof given by Gell-Mann, Goldberger and Thirring was based on the use of perturbation theory for the electromagnetic interaction. This deficiency was remedied by Goldberger [10] soon after.

The problem of dispersion relations for the forward scattering of particles with mass had been touched on in a heuristic way by Gell-Mann, Goldberger and Thirring. Goldberger [11] attacked the problem again, basing his discussion on the microscopic causality condition mentioned above and a form of the scattering amplitude derived independently by Low [12] and Lehmann, Symanzik and Zimmermann.[13] In the meantime Goldberger and others [14] applied the dispersion relations to the forward scattering of pions by nucleons. The first real comparison of theory and experiment was made by Anderson, Davidon and Kruse.[15] Since that time various comparisons of theory and experiment have been made for different purposes—to deduce a value for the pion-nucleon coupling constant,[16] to distinguish between different sets of phase shifts,[17] to relate *s*-wave and *p*-wave phase shifts.[18] A number of these problems will be dealt with in detail later.

The next theoretical step taken was the generalisation from forward scattering to non-forward scattering. Heuristic derivations were given independently by Gell-Mann and Polkinghorne, and Goldberger, Nambu and Oehme (unpublished), Salam [19] and Capps and Takeda.[20] The dispersion relations so obtained were in the energy variable of the problem, with the Lorentz invariant momentum transfer held constant. These relations were applied to pion-nucleon scattering by Chew, Goldberger, Low and Nambu.[21] The same authors [22] derived equivalent relations for photoproduction of pions from nucleons and discussed the photoproduction process for energies up to and including the (3,3) resonance. Similar dispersion relations for photoproduction were presented at the same time by Corinaldesi,[23] and by Logunov and Stepanov.[24] Since then several groups of workers have made comparisons with photoproduction data. Some of these aspects will be mentioned below.

At the same time that the dispersion relations were being put to work, rigorous proofs were being given to establish the relations on a firm theoretical foundation. For forward scattering a rigorous proof was given by Symanzik.[25] The best known general proofs, in order of appearance and of decreasing complexity, are those of Bogoliubov, Medvedev and Polivanov,[26] Bremermann, Oehme and Taylor,[27] and

Lehmann.[28] These proofs showed that the fixed momentum transfer dispersion relations were valid provided that the momentum transfer was less than some maximum and that certain inequalities on the masses of the particles were satisfied. Lehmann proved in addition that the scattering amplitude had a certain domain of analyticity in the momentum transfer variable for fixed energy. In the non-relativistic domain, Khuri [29] gave a rigorous proof of fixed momentum transfer dispersion relations for the scattering of a particle by a fixed ordinary potential, using the methods of Fredholm integral equations. His work was later extended by Klein and Zemach.[30]

Once the obvious derivations and applications to the pion-nucleon problem and photon interactions had been made, the theoretical physicists were forced to look for other areas of application. In the area of the strong interactions of ordinary particles, work was done on problems such as pion-deuteron scattering,[31] nucleon-nucleon scattering,[32] electromagnetic form factors of the nucleons.[33]

Dispersion relations were also written down for strange particle interactions, particularly for *K*-meson-nucleon scattering.[34] An attempt was made to deduce the parity of the *K* meson by means of these dispersion relations, but the presence of an unphysical region in the dispersion integrals and the lack of adequate experimental data prevented any conclusions.

The other area where dispersion relation techniques were exploited was that of decay processes involving strong interactions as well as weak. Corinaldesi [35] wrote down dispersion relations for decay processes and later applied them to tau decay. Bogoliubov, Logunov and Bilenky [36] presented a general discussion of dispersion relations for weak interactions. Goldberger and Treiman, in a series of papers, used dispersion relation techniques to consider the effects of the strong interactions in the decay of the pion,[37] in beta decay and mu-meson capture,[38] and the decay of the neutral pion.[39] Other decay processes have been considered by others subsequently.

3. Present Era

Although applications of the type of dispersion relations described so far (one variable, usually energy, " dispersed ") still continue to be made, the emphasis has shifted to the so-called double dispersion relations (or Mandelstam representation). The activities divide themselves into two areas: (i) the use of the Mandelstam representation for discussion of physical problems and (ii) the investigation of its validity or, more generally, of the analytic properties of amplitudes as functions of two or more independent variables.

The double dispersion relations were first conjectured by Mandelstam[40] in 1958 in connection with pion-nucleon scattering. Since Mandelstam's conjecture attributed certain analytic properties to the amplitude as a function not only of the complex energy variable, but also of the complex momentum transfer variable, it provided a much more powerful handle on the problem than the ordinary dispersion relations. In fact, the proposal was that the double dispersion relations, together with unitarity, might be a framework for complete calculations involving a minimum number of empirical parameters (coupling constants). The elaboration of this proposal and its realisation in a practical scheme of calculation will be discussed in detail by Prof. G. F. Chew.

The Mandelstam conjecture aroused a great deal of interest in the general analytic properties of scattering amplitudes and vertex functions. There had, of course, always been interest in these aspects by a selected group of theorists. The rigorous work of Källén and Wightman [41] on the explicit domain of analyticity of the vertex function is one example. The Jost-Lehmann-Dyson representation of the Fourier transform of a retarded commutator is another.[42] But there have been many investigations based on perturbation theory. The analytic properties of the vertex function were examined in perturbation theory by Karplus, Sommerfield, and Wichmann.[43] Mandelstam [44] considered the two particles scattering amplitude and established the validity of his representation to fourth order, and, in the one-meson approximation, to all orders. Landau [45] and Taylor [46] independently treated the problem of determining the singularities of an arbitrary Feynman amplitude. Dr Polkinghorne will be discussing these papers in detail, as well as his own work on the subject.[47]

Closely related to the general work on analytic properties, but with definite practical applications in mind, is the problem of the singularities of the S-matrix lying closest to the physical domain. Chew first made plausible that, in a scattering process where the lowest order diagram corresponds to the exchange of a particle of mass m with momentum transfer q, the scattering amplitude would have a pole at unphysical angles such that $m^2+q^2 = 0$. Furthermore, the residue of the pole would be the appropriate renormalized coupling constant. Chew [48] then suggested an extrapolation procedure for obtaining the pion-nucleon coupling constant from nucleon-nucleon scattering. The procedure was generalised to problems with more than two particles in the final state by Chew and Low [49] and to general two body reactions by Taylor.[50] With certain assumptions about the analytic properties of the vertex functions involved, a proof of the presence of the conjectured poles has been given.[51] Moravcsik, Taylor and others have made numerous applications of the extrapolation approach to the determination of parities and coupling constants. A natural extension of this idea is to use the properties (if known) of the nearest singularities to write the appropriate amplitude as a sum of known contributions from the nearby regions of the complex plane and unknown (or less well known) contributions from farther away. Then the distant terms can be calculated approximately or treated phenomenologically. All these techniques will be discussed by Prof. Moravcsik.

Other ramifications of the Mandelstam conjecture are the proofs of the domain of analyticity of the scattering amplitude for nonrelativistic scattering,[52] the development of dispersion relations for individual partial waves in a scattering process,[53] and the use of the Mandelstam representation to explore the influence of the pion-pion interaction on electromagnetic form factors.[54] Some of these topics will be covered in detail by Prof. Frazer.

2

Elementary Considerations

1. Classical Kronig-Kramers Dispersion Relations

The fundamental relation deduced by Kronig and Kramers from the classical Lorentz theory of dispersion of light connects the real part of the index of refraction to the absorption coefficient. We define the complex index of refraction,

$$n(\omega) = n_r(\omega) + \frac{ic}{2\omega}\alpha(\omega) \quad \text{.........}(2.1)$$

where $\alpha(\omega)$ is the absorption coefficient. Then the Kronig-Kramers dispersion relation is

$$n_r(\omega) = 1 + \frac{c}{\pi} P \int_0^\infty \frac{\alpha(\omega')}{\omega'^2 - \omega^2} d\omega' \quad \text{.........}(2.2)$$

where P means Cauchy principal value.

Kramers observed that $n(\omega)$ was defined only for positive frequencies. He then defined its value for negative frequencies by $n(-\omega) = n^*(\omega)$ (now known as crossing symmetry). This means that $n_r(\omega)$ and $\alpha(\omega)$ are assumed even functions of ω. With this extension to negative, real frequencies, (*2.2*) can be written

$$n_r(\omega) = 1 + \frac{1}{\pi} P \int_{-\infty}^\infty \frac{c\alpha(\omega')/2\omega'}{\omega' - \omega} d\omega', \quad \text{.........}(2.3)$$

or, in terms of real and imaginary parts of $n(\omega)$,

$$\text{Re}\,[n(\omega) - 1] = \frac{1}{\pi} P \int_{-\infty}^\infty \frac{\text{Im}\,[n(\omega') - 1]}{\omega' - \omega} d\omega'. \quad \text{.........}(2.4)$$