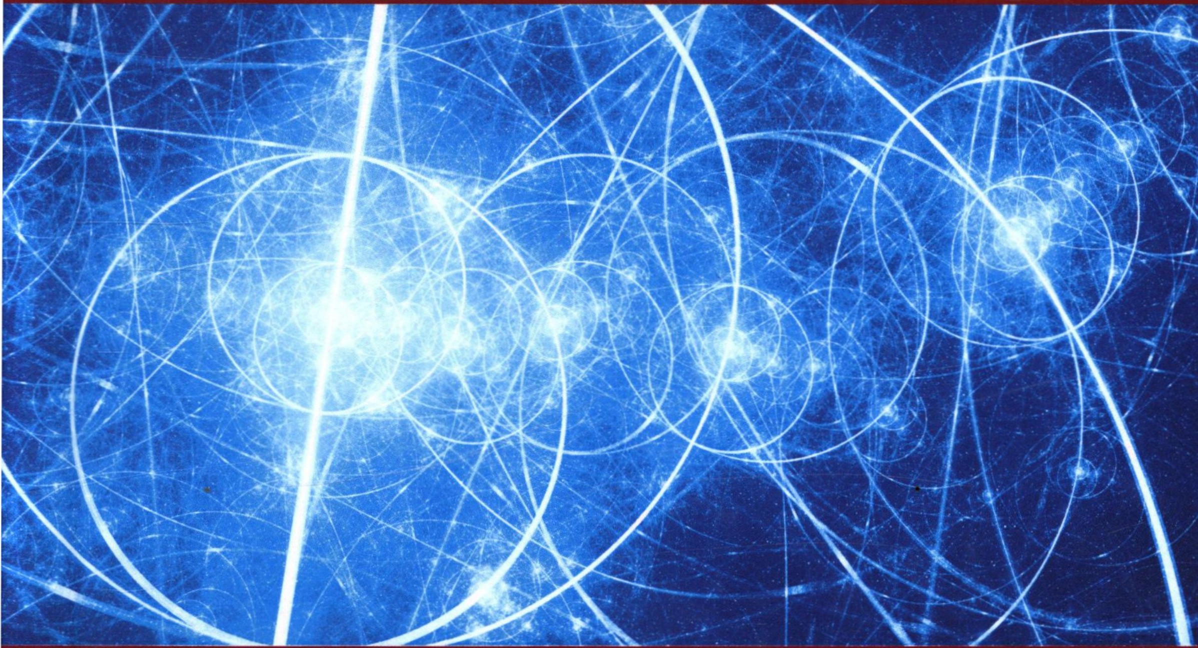
MATHEMATICS AND STATISTICS



Theory and Statistical Applications of Stochastic Processes

Yuliya Mishura
Georgiy Shevchenko

ISTE WILEY

This book is concerned with the theory of stochastic processes and the theoretical aspects of statistics for stochastic processes. It combines classic topics such as construction of stochastic processes, associated filtrations, processes with independent increments, Gaussian processes, martingales, Markov properties, continuity and related properties of trajectories with contemporary subjects: integration with respect to Gaussian processes, Itô integration, stochastic analysis, stochastic differential equations, fractional Brownian motion and parameter estimation in diffusion models.

The presentation is made as self-contained as possible, with complete proofs of the facts which are often either omitted from textbooks or are replaced by informal or heuristic arguments. Some auxiliary material, related mainly to different subjects of real analysis and probability theory, is included in the comprehensive appendix. The book is targeted at the widest audience: students of mathematical and related programs, postgraduate students, postdocs, lecturers, researchers and practitioners in any field concerned with the application of stochastic processes will find this book to be a valuable resource.

Yuliya Mishura is Full Professor and Head of the Department of Probability Theory, Statistics and Actuarial Mathematics at Taras Shevchenko National University of Kyiv. Her research interests include martingale theory, fractional processes, stochastic differential equations, the statistics of stochastic processes and financial mathematics.

Georgiy Shevchenko is Full Professor in the Department of Probability Theory, Statistics and Actuarial Mathematics at Taras Shevchenko National University of Kyiv. His research interests include stochastic analysis, stochastic differential equations, models with long memory, the statistics of stochastic processes and financial mathematics.

iSTE
www.iste.co.uk

WILEY

9 781786 300508

Yuliya Mishura
Georgiy Shevchenko

Theory and Statistical Applications of Stochastic Processes

WILEY
iSTE

Series Editor
Nikolaos Limnios

Theory and Statistical Applications of Stochastic Processes

Yuliya Mishura
Georgiy Shevchenko

iSTE

WILEY

First published 2017 in Great Britain and the United States by ISTE Ltd and John Wiley & Sons, Inc.

ISTE Ltd
27-37 St George's Road
London SW19 4EU
UK

www.iste.co.uk

John Wiley & Sons, Inc.
111 River Street
Hoboken, NJ 07030
USA

www.wiley.com

Library of Congress Control Number: 2017953309

British Library Cataloguing-in-Publication Data
A CIP record for this book is available from the British Library
ISBN 978-1-78630-050-8

MIX
Paper from responsible sources
FSC® C013604
FSC www.fsc.org

Theory and Statistical Applications of Stochastic Processes

Preface

This book is concerned with both mathematical theory of stochastic processes and some theoretical aspects of statistics for stochastic processes. Our general idea was to combine classic topics of the theory of stochastic processes – measure-theoretic issues of existence, processes with independent increments, Gaussian processes, martingales, continuity and related properties of trajectories and Markov properties – with contemporary subjects – stochastic analysis, stochastic differential equations, fractional Brownian motion and parameter estimation in diffusion models. A more detailed exposition of the contents of the book is given in the Introduction.

We aimed to make the presentation of material as self-contained as possible. With this in mind, we have included several complete proofs, which are often either omitted from textbooks on stochastic processes or replaced by some informal or heuristic arguments. For this reason, we have also included some auxiliary materials, mainly related to different subjects of real analysis and probability theory, in the comprehensive appendix. However, we could not cover the full scope of the topic, so a substantial background in calculus, measure theory and probability theory is required.

The book is based on lecture courses, *Theory of stochastic processes*, *Statistics of stochastic processes*, *Stochastic analysis*, *Stochastic differential equations*, *Theory of Markov processes*, *Generalized processes of fractional Brownian motion and Diffusion processes*, taught regularly in the Mechanics and Mathematics Faculty of Taras Shevchenko National University of Kyiv and *Stochastic differential equations* lecture courses taught at the University of Verona in Spring 2016; *Fractional Brownian motion and related processes: stochastic calculus, statistical applications and modeling* taught in School in Bedlewo in March 2015; *Fractional Brownian motion and related processes* taught at Ulm University in June 2015; and a *Fractional Brownian motion in a nutshell* mini-course given at the 7th Jagna International Conference in 2014.

The book is targeted at the widest audience: students of mathematical and related programs, postgraduate students, postdoctoral researchers, lecturers, researchers, practitioners in the fields concerned with the application of stochastic processes, etc. The book would be most useful when accompanied by a problem in stochastic processes; we recommend [GUS 10] as it matches our topics best.

We would like to express our gratitude to everyone who made the creation of this book possible. In particular, we would like to thank Łukasz Stettner, Professor at the Department of Probability Theory and Mathematics of Finance, Institute of Mathematics, Polish Academy of Sciences; Luca Di Persio, Assistant Professor at the Department of Computer Science at the University of Verona; Evgeny Spodarev, Professor and Director of the Institute of Stochastics at Ulm University, for their hospitality while hosting Yuliya Mishura during lecture courses. We would also like to thank Alexander Kukush, Professor at the Department of Mathematical Analysis of Taras Shevchenko National University of Kyiv, for proofreading the statistical part of the manuscript, and Evgeniya Munchak, PhD student at the Department of Probability, Statistics, and Actuarial Mathematics of Taras Shevchenko National University of Kyiv, for her help in typesetting the manuscript.

Yuliya MISHURA
Georgiy SHEVCHENKO

September 2017

Introduction

In the world that surrounds us, a lot of events have a random (nondeterministic) structure. At molecular and subatomic levels, all natural phenomena are random. Movement of particles in the surrounding environment is accidental. Numerical characteristics of cosmic radiation and the results of monitoring the effect of ionizing radiation are random. The majority of economic factors surrounding asset prices on financial markets vary randomly. Despite efforts to mitigate risk and randomness, they cannot be completely eliminated. Moreover, in complex systems, it is often easier to reach an equilibrium state when they are not too tightly controlled. Summing-up, chance manifests itself in almost everything that surrounds us, and these manifestations vary over time. Anyone can simulate time-varying randomness by tossing a coin or rolling a dice repeatedly and recording the results of successive experiments. (If a physical random number is unavailable, one of the numerous computer algorithms to generate random numbers can be used.) In view of this ubiquity of randomness, the theory of probability and stochastic processes has a long history, despite the fact that the rigorous mathematical notion of probability was introduced less than a century ago. Let us speak more on this history.

People have perceived randomness since ancient times, for example, gambling already existed in ancient Egypt before 3000 BC. It is difficult to tell exactly when systematic attempts to understand randomness began. Probably, the most notable were those made by the prominent ancient Greek philosopher Epicurus (341–270 BC). Although his views were heavily influenced by Democritus, he attacked Democritus' materialism, which was fully deterministic. Epicurus insisted that all atoms experience some random perturbations in their dynamics. Although modern physics confirms these ideas, Epicurus himself attributed the randomness to the free will of atoms. The phenomenon of random detours of atoms was called *clinamen* (cognate to inclination) by the Roman poet Lucretius, who had brilliantly exposed Epicurus' philosophy in his poem *On the Nature of Things*.

Moving closer to present times, let us speak of the times where there was no theory of stochastic processes, physics was already a well-developed subject, but there wasn't any equipment suitable to study objects in sufficiently small microscopic detail. In 1825, botanist Robert Brown first observed a phenomenon, later called *Brownian motion*, which consisted of a chaotic movement of a pollen particle in a vessel. He could not come up with a model of this system, so just stated that the behavior is random.

A suitable model for the phenomenon arose only several decades later, in a very different problem, concerned with the pricing of financial assets traded on a stock exchange. A French mathematician Louis Bachelier (1870–1946), who aimed to find a mathematical description of stochastic fluctuations of stock prices, provided a mathematical model in his thesis "Théorie de la spéculation" [BAC 95], which was defended at the University of Paris in 1900. The model is, in modern terms, a stochastic process, which is characterized by the fact that its increments in time, in a certain statistical sense, are proportional to the square root of the time change; this "square root" phenomenon had also be observed earlier in physics; Bachelier was the first to provide a model for it. Loosely speaking, according to Bachelier, the asset price S_t at time t is modeled by

$$S_t = at + b\sqrt{t}\xi,$$

where a, b are constant coefficients, and ξ is a random variable having Gaussian distribution.

The work of Bachelier was undervalued, probably due to the fact that applied mathematics was virtually absent at the time, as well as concise probability theory. Bachelier spent his further life teaching in different universities in France and never returned to the topic of his thesis. It was only brought to the spotlight 50 years after its publication, after the death of Bachelier. Now, Bachelier is considered a precursor of mathematical finance, and the principal organization in this subject bears his name: Bachelier Finance Society.

Other works which furthered understanding towards Brownian motion were made by prominent physicists, Albert Einstein (1879–1955) and Marian Smoluchowski (1872–1917). Their articles [EIN 05] and [VON 06] explained the phenomenon of Brownian motion by thermal motion of atoms and molecules. According to this theory, the molecules of a gas are constantly moving with different speeds in different directions. If we put a particle, say of pollen which has a small surface area, inside the gas, then the forces from impacts with different molecules do not compensate each other. As a result, this *Brownian* particle will experience a chaotic movement with velocity and direction changing approximately 10^{14} times per second. This gave a physical explanation to the phenomenon observed by the botanist. It also turned out that a kinetic theory of thermal motion required a stochastic process B_t. Einstein and Smoluchowski not only described this stochastic process, but also found its important probabilistic characteristics.

Only a quarter of a century later, in 1931, Andrey Kolmogorov (1903–1987) laid the groundwork for probability theory in his pioneering works *About the Analytical Methods of Probability Theory* and *Foundations of the Theory of Probability* [KOL 31, KOL 77]. This allowed his fellow researcher Aleksandr Khinchin (1894–1859) to give a definition of stochastic process in his article [KHI 34].

There is an anecdote related to the role of Khinchin in defining a stochastic process and the origins of the "stochastic" as a synonym for randomness (the original Greek word means "guessing" and "predicting"). They say that when Khinchin defined the term "random process", it did not go well with the Soviet authorities. The reason is that the notion of random process used by Khinchin contradicted dialectical materialism (diamat). In diamat, similarly to Democritus' materialism, all processes in nature are characterized by totally deterministic development, transformation, etc., so the phrase "random process" itself sounded paradoxical. As a result, to avoid dire consequences (we recall that 1934 was the apogee of Stalin's Great Terror), Khinchin had to change the name. After some research, he came up with the term "stochastic", from *στοχαστική τέχνη*, the Greek title of *Ars Conjectandi*, a celebrated book by Jacob Bernoulli (1655–1705) published in 1713, which contains many classic results. Being popularized later by William Feller [FEL 49] and Joseph Doob [DOO 53], this became a standard notion in English and German literature. Perhaps paradoxically, in Russian literature, the term "stochastic processes" did not live for long. The 1956 Russian translation of Doob's monograph [DOO 53] of this name was entitled *Probabilistic processes*, and now the standard name is *random process*.

An alternative explanation, given, for example, in [DEL 17], attributes the term "stochastic" to Ladislaus Władysław Bortkiewicz (1868–1931), Russian economist and statistician, who in his paper, *Die Iterationen* [BOR 17], defined the term "stochastic" as "the investigation of empirical varieties, which is based on probability theory, and, therefore, on the law of large numbers. But stochastic is not simply probability theory, but above all probability theory and applications". This meaning correlates with the one given in *Ars Conjectandi* by Jacob Bernoulli, so the true origin of the term probably is somewhere between these two stories. It is also worth mentioning that Bortkiewicz is known for proving the *Poisson approximation theorem* about the convergence of binomial distributions with small parameters to the Poisson distribution, which he called *the law of small numbers*.

This historical discussion would be incomplete without mentioning Paul Lévy (1886–1971), a French mathematician who made many important contributions to the theory of stochastic processes. Many objects and theorems now bear his name: *Lévy processes*, *Lévy-Khinchin representation*, *Lévy representation*, etc. Among other things, he wrote the first extensive monograph on the (mathematical model of) Brownian motion [LÉV 65].

Further important progress in probability theory is related to Norbert Wiener (1894–1964). He was a jack of all trades: a philosopher, a journalist, but the most important legacy that he left was as a mathematician. In mathematics, his interest was very broad, from number theory and real analysis, to probability theory and statistics. Besides many other important contributions, he defined an integral (of a deterministic function) with respect to the mathematical model of Brownian motion, which now bears his name: a *Wiener process* (and the corresponding integral is called a *Wiener integral*).

The ideas of Wiener were developed by Kiyoshi Itô (1915–2008), who introduced an integral of random functions with respect to the Wiener process in [ITÔ 44]. This lead to the emergence of a broad field of *stochastic analysis*, a probabilistic counterpart to real integro-differential calculus. In particular, he defined *stochastic differential equations* (the name is self-explanatory), which allowed us to study *diffusion processes*, which are natural generalizations of the Wiener process. As with Lévy, many objects in stochastic analysis are named after Itô: *Itô integral*, *Itô process*, *Itô representation*, *Wiener-Itô decomposition*, etc.

An important contribution to the theory of stochastic processes and stochastic differential equations was made by Ukrainian mathematicians Iosif Gihman (1918–1985) and more notably by Anatoliy Skorokhod (1930–2011). Their books [GIH 72, GIK 04a, GIK 04b, GIK 07] are now classical monographs. There are many things in stochastic analysis named after Skorokhod: *Skorokhod integral*, *Skorokhod space*, *Skorokhod representation*, etc.

Our book, of course, is not the first book on stochastic processes. They are described in many other texts, from some of which we have borrowed many ideas presented here, and we are grateful to their authors for the texts. It is impossible to mention every single book here, so we cite only few texts of our selection. We apologize to the authors of many other wonderful texts which we are not able to cite here.

The extensive treatment of probability theory with all necessary context is available in the books of P. Billingsley [BIL 95], K.-L. Chung [CHU 79], O. Kallenberg [KAL 02], L. Koralov and Y. Sinai [KOR 07], M. Loève [LOÈ 77, LOÈ 78], D. Williams [WIL 91]. It is also worth mentioning the classic monograph of P. Billingsley [BIL 99] concerned with different kinds of convergence concepts in probability theory.

For books which describe the theory of stochastic processes in general, we recommend that the reader looks at the monograph by J. Doob [DOO 53], the extensive three-volume monograph by I. Gikhman and A. Skorokhod

[GIK 04a, GIK 04b, GIK 07], the textbooks of Z. Brzezniak and T. Zastawniak [BRZ 99], K.-L. Chung [CHU 79], G. Lawler [LAW 06], S. Resnick [RES 92], S. Ross [ROS 96], R. Schilling and L. Partzsch [SCH 14], A. Skorokhod [SKO 65], J. Zabczyk [ZAB 04]. It is also worth mentioning the book by A. Bulinskiy and A. Shiryaev [BUL 05], from which we borrowed many ideas; unfortunately, it is only available in Russian. Martingale theory is well presented in the books of R. Liptser and A. Shiryaev [LIP 89], J. Jacod and A. Shiryaev [JAC 03], L. Rogers and D. Williams [ROG 00a], and the classic monograph of D. Revuz and M. Yor [REV 99]. There are many excellent texts related to different aspects of Lévy processes, including the books of D. Applebaum [APP 09], K. Sato [SAT 13], W. Schoutens [SCH 03], and the collection [BAR 01].

Stochastic analysis now stands as an independent subject, so there are many books covering different aspects of it. The books of K.-L. Chung and D. Williams [CHU 90], I. Karatzas and S. Shreve [KAR 91], H. McKean [MCK 69], J.-F. Le Gall [LEG 16], L. Rogers and D. Williams [ROG 00b] cover stochastic analysis in general, and the monograph of P. Protter [PRO 04] goes much deeper into integration issues. Stochastic differential equations and diffusion processes are the subject of the best-selling textbook of B. Øksendal [ØKS 03], and the monographs of N. Ikeda and S. Watanabe [IKE 89], K. Itô and H. McKean [ITÔ 74], A. Skorokhod [SKO 65], and D. Strook and S. Varadhan [STR 06]. The ultimate guide to Malliavin calculus is given by D. Nualart [NUA 06]. Concerning financial applications, stochastic analysis is presented in the books of T. Björk [BJÖ 04], M. Jeanblanc, M. Yor, and M. Chesney [JEA 09], A. Shiryaev [SHI 99], and S. Shreve [SHR 04].

Different aspects of statistical methods for stochastic processes are covered by the books of P. Brockwell and R. Davis [BRO 06], C. Heyde [HEY 97], Y. Kutoyants [KUT 04], G. Seber and A. Lee [SEB 03].

Finally, fractional Brownian motion, one of the main research interests of the authors of this book, is covered by the books of F. Biagini *et al.* [BIA 08], Y. Mishura [MIS 08], I. Nourdin [NOU 12], D. Nualart [NUA 06], and by lecture notes of G. Shevchenko [SHE 15].

Our book consists of two parts: the first is concerned with the theory of stochastic processes and the second with statistical aspects.

In the first chapter, we define the main subjects: stochastic process, trajectory and finite-dimensional distributions. We discuss the fundamental issues: existence and construction of a stochastic process, measurability and other essential properties, and sigma-algebras generated by stochastic processes.

The second chapter is devoted to stochastic processes with independent increments. A definition is given and simple criteria which provide the existence are discussed. We also provide numerous important examples of processes with independent increments, including Lévy processes, and study their properties.

The third chapter is concerned with a subclass of stochastic processes, arguably the most important for applications: Gaussian processes. First, we discuss Gaussian random variables and vectors, and then we give a definition of Gaussian processes. Furthermore, we give several important examples of Gaussian processes and discuss their properties. Then, we discuss integration with respect to Gaussian processes and related topics. Particular attention is given to fractional Brownian motion and Wiener processes, with discussion of several integral representations of fractional Brownian motion.

The fourth chapter focuses on some delicate properties of two Gaussian processes, which are of particular interest for applications: the Wiener process and fractional Brownian motion. In particular, an explicit construction of the Wiener process is provided and nowhere differentiability of its trajectories is shown. Having in mind the question of parameter estimation for stochastic processes, we also discuss the asymptotic behavior of power variations for the Wiener process and fractional Brownian motion in this chapter.

In the fifth chapter, we attempted to cover the main topics in the martingale theory. The main focus is on the discrete time case; however, we also give several results for stochastic processes. In particular, we discuss the notions of stochastic basis with filtration and stopping times, limit behavior of martingales, optional stopping theorem, Doob decomposition, quadratic variations, maximal inequalities by Doob and Burkholder-Davis-Gundy, and the strong law of large numbers.

The sixth chapter is devoted to properties of trajectories of a stochastic process. We introduce different notions of continuity as well as important concepts of separability, indistinguishability and stochastic equivalence, and establish several sufficient conditions for continuity of trajectories and for absence of discontinuities of the second kind. To the best of our knowledge, this is the first time that the different aspects of regularity and continuity are comprehensively discussed and compared.

The seventh chapter discusses Markov processes. The definition, together with several important examples, is followed by analytical theory of Markov semigroups. The chapter is concluded by the investigation of diffusion processes, which serves as a bridge to stochastic analysis discussed in the following chapters. We provide a definition and establish important criteria and characterization of diffusion processes. We pay particular attention to the forward and backward Kolmogorov equations, which are of great importance for applications.

In the eighth chapter, we give the classical introduction to stochastic integration theory, which includes the definition and properties of Itô integral, Itô formula, multivariate stochastic calculus, maximal inequalities for stochastic integrals, Girsanov theorem and Itô representation.

The ninth chapter, which closes the theoretical part of the book, is concerned with stochastic differential equations. We give a definition of stochastic differential equations and establish the existence and uniqueness of its solution. Several properties of the solution are established, including integrability, continuous dependence of the solution on the initial data and on the coefficients of the equation. Furthermore, we prove that solutions to stochastic differential equations are diffusion processes and provide a link to partial differential equations, the Feynman-Kac formula. Finally, we discuss the diffusion model of a financial market, giving notions of arbitrage, equivalent martingale measure, pricing and hedging of contingent claims.

The tenth chapter opens the second part of the book, which is devoted to statistical aspects. It studies the estimation of parameters of stochastic processes in different scenarios: in a linear regression model with discrete time, in a continuous time linear model driven by Wiener process, in models with fractional Brownian motions, in a linear autoregressive model and in homogeneous diffusion models.

In the eleventh chapter, the classic problem of optimal filtering is studied. A statistical setting is described, then a representation of optimal filter is given as an integral with respect to an observable process. Finally, the integral Wiener-Hopf equation is derived, a linear stochastic differential equation for the optimal filter is derived, and the error of the optimal filter is identified in terms of solution of the Riccati equation. In the case of constant coefficients, the explicit solutions of these equations are found.

Auxiliary results, which are referred to in the book, are collected in Appendices 1 and 2. In Appendix 1, we give essential facts from calculus, measure theory and theory of operators. Appendix 2 contains important facts from probability theory.