Convex Optimization for Signal Processing and Communications

From Fundamentals to Applications

Chong-Yung Chi · Wei-Chiang Li · Chia-Hsiang Lin

CRC Press
Taylor & Francis Group

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CRC Press
Taylor & Francis Group
Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group, an **Informa** business

CRC Press
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

Printed on acid-free paper
Version Date: 20161215

International Standard Book Number-13: 978-1-4987-7645-5 (Hardback)

Visit the Taylor & Francis Web site at
http://www.taylorandfrancis.com

and the CRC Press Web site at
http://www.crcpress.com

Printed and bound in the United States of America by
Edwards Brothers Malloy on sustainably sourced paper

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Preface

Convex Optimization has been recognized as a powerful tool for solving many science and engineering problems. Over the last two decades, convex optimization has been successfully and extensively applied to various problems in signal processing such as blind source separation (BSS) for biomedical and hyperspectral image analysis, and in multiple-input multiple-output (MIMO) wireless communications and networking, such as coherent/noncoherent detection, transmit/robust/distributed beamforming, and physical-layer secret communications. Particularly, fourth generation (4G) wireless communication systems have been in operation, and various researches for fifth generation (5G) systems, e.g., massive MIMO, millimeter wave wireless communications, full-duplex MIMO, energy harvesting, and multicell coordinated beamforming, have been intensively studied and reported in the open literature, where the convex optimization tool is extensively wielded, validating its central role to the development of 5G systems and to many interdisciplinary science and engineering applications.

Next, let us address the motivation, organization of the book, suggestions for instructors, and acknowledgment of writing the book *"Convex Optimization for Signal processing and Communication: From Fundamentals to Applications,"* respectively.

Motivation:

Since Spring 2008, I have been teaching the graduate-level course "*Optimization for Communications*" at National Tsing Hua University (NTHU), Hsinchu, Taiwan. As in teaching any other course, I prepared my own lecture notes for this course. My lecture notes are primarily based on the seminal textbook, *Convex Optimization* (by Stephen Boyd and Lieven Vandenberghe), Cambridge University Press, 2004; some research results published in the open literature; and some materials offered by my former colleague, Prof. Wing-Kin Ma (Chinese University of Hong Kong), who taught this course at NTHU from August 2005 through July 2007.

From my teaching experience, many engineering students are often at a loss in abstract mathematics due to lack of tangible linkage between mathematical theory and applications. Consequently they would gradually lose the motivation of learning powerful mathematical theory and tools, thereby leading to losing momentum to solve research problems by using the mathematics they were trying to learn. In order to help students to be fully equipped with this powerful tool, my lecture notes on convex optimization are molded into a bridge from the fundamental mathematical theory to practical applications. I have assembled my lecture notes in this book, and sincerely hope that the readers, especially the student community, will benefit from the materials presented here.

Over the last decade, my lecture notes have been successfully used 12 times for my intensive 2-week short course *"Convex Optimization for Signal Processing and Communications"* at major universities in China, including Shandong University (January 2010), Tsinghua University (August 2010 and August 2012), Tianjin University (August 2011), Beijing Jiaotong University (July 2013 and July 2015), University of Electronic Science and Technology of China (November 2013, September 2014, and September 2015), Xiamen University (December 2013), Sun Yat-Sen University (August 2015), and Beijing University of Posts and Telecommunications (July 2016). These short courses differed from traditional short courses in conferences, workshops, and symposia (usually using a set of synoptic slides without enough details due to limited time). In each short course I offered in China, I spent around 32 lecture hours over two consecutive weeks, going through almost all the theories, proofs, illustrative examples, algorithm design and implementation, and some state-of-the-art research applications in detail, like a guided journey/exploration from fundamental mathematics to cutting-edge researches and applications rather than pure mathematics. Finally, a post term project was offered for the attendees to get hands-on experience of solving some advisable problems afterwards. I have received many positive feedbacks from the short-course attendees, and now many of them are good at using convex optimization in solving research problems, leading to many research breakthroughs and successful applications.

Organization of the book:

With a balance between mathematical theory and applications, this book provides an introduction to convex optimization from fundamentals to applications. It is suitable for the first-year graduate course "*Convex Optimization*" or "*Nonlinear Optimization*" for engineering students who need to solve optimization problems, and meanwhile wish to clearly see the link between mathematics and applications in hands. Some mathematical prerequisites such as linear algebra, matrix theory, and calculus are surely much help in reading this book.

The book contains 10 chapters and an appendix, basically written in a causally sequential fashion; namely, to have in-depth learning in each chapter, one needs to absorb the materials introduced in early chapters. Chapter 1 provides some mathematical background materials that will be used in the ensuing chapters. Chapter 2 introduces convex sets and Chapter 3 introduces convex functions that are essential to the subsequent introduction of convex problems and problem reformulations in Chapter 4, along with many examples in each of the these chapters.

Some widely known convex optimization problems (or simply termed as convex problems) are introduced next, including geometric programming (GP) that is introduced in Chapter 5 (where a geometric program, nonconvex at first glance, can be easily reformulated into a convex problem); linear programming (LP), quadratic programming (QP) and quadratically constrained quadratic programming (QCQP) that are introduced in Chapter 6; second-order cone programming (SOCP) that is introduced in Chapter 7; and semidefinite programming (SDP) that is introduced in Chapter 8. Each of these chapters presents how the essential materials (introduced in Chapter 2 to 4) are advisably and effectively applied to practical problems in communications and/or signal processing. However, we only present key ideas, philosophies, and major reformulations for solving the problem under consideration. Some simulation results, also real data experiments (in biomedical and hyperspectral image analysis), are presented for the readers to visually see the solution accuracy and efficiency of the designed algorithms. Readers can refer to the associated research papers for full details to ascertain whether he/she can understand/apply the convex optimization theory comprehensively. Because SDP has been extensively used in wireless communications and networking, we especially introduce more challenging applications in Chapter 8, where various intricate optimization problems involving SDP have been prevalent in the evolution towards 5G.

In Chapter 9, we introduce "*duality*" which is of paramount importance and a perfect complement to Chapters 2 through 4, because some convex problems can be solved more efficiently by using Karush–Kuhn–Tucker (KKT) conditions introduced in Chapter 9, comparing using the optimality conditions introduced in the early chapters, and vice versa. In our experience, analytical performance evaluation and complexity analysis of the designed algorithm for solving an optimization problem is crucial not only to the algorithm design in a perspective and insightful manner, but also to the future direction/clues for further research breakthroughs. These analyses can justify and interpret the simulation and experimental results qualitatively and quantitatively, thereby providing a concrete foundation for the applicability of the designed algorithm. However, these analyses heavily rely on the delicate duality theory. On the other hand, once an optimization problem is reformulated into a convex problem, it can be readily solved by using off-the-shelf convex solvers, e.g., CVX and SeDuMi, which are briefly introduced in the Appendix. This may be adequate during the research stage, but not necessarily suitable for practical applications, where real-time processing or on-line processing is highly desired or required. Chapter 10 introduces the interior-point method that actually tries to numerically solve the KKT conditions introduced in Chapter 9, which has been widely used for the realization of obtaining a solution of a specific convex problem in a more computationally efficient manner.

Suggestions for instructors:

For instructors who consider teaching this subject with this book for a one-semester course, I have a few suggestions based on my years of teaching experience. First of all, Chapter 1 through 4 can be covered followed by a midterm examination. Next, some selected applications in Chapter 5 to 8 can be covered, and then a term project of studying a research paper can be announced. The purpose of the term project is for students (1 to 2 students as a group) to experience how a practical problem can be solved by using what they have learned to verify all the theory, analysis and simulation/experimental results of the assigned paper. Then the instructor can continue to teach Chapter 9 and 10. Finally, students would take the final examination, followed by an oral presentation from each term project group. After several implementations myself, I found this practice quite inspirational and beneficial to students.

Acknowledgment:

Over the last eight years' accumulation of my lecture notes, this book was accomplished through tremendous voluntary efforts from many of my former students, including my former PhD students, Dr. ArulMurugan Ambikapathi, Dr. Kun-Yu Wang, Dr. Wei-Chiang Li, and Dr. Chia-Hsiang Lin and my former Master students, Yi-Lin Chiou, Yu-Shiuan Shen, Tung-Chi Ye and Yu-Ping Chang who helped draw many figures in the book. I would also like to thank my former colleague, Prof. Wing-Kin Ma, my former PhD students, Dr. Tsung-Hui Chang and Dr. Tsung-Han Chan, and former visiting scholar, Dr. Fei Ma, and former visiting PhD students, Dr. Xiang Chen, Dr. Chao Shen, Dr. Haohao Qin, Dr. Fei He, Gui-Xian Xu, Kai Zhang, Yang Lu, Christian Weiss, and visiting Master students Lei Li and Ze-Liang Ou, and my PhD student Yao-Rong Syu and Master student Amin Jalili, and all of my graduate students who have offered voluntary assistance, either directly or indirectly.

I would like particularly to express my deep appreciation to those participants of my short courses offered in the above-mentioned major universities in Mainland China over the last seven years, for their numerous questions, interactions, and comments that have been taken into account during the writing of this book, thereby significantly improving the readability of the book, especially to engineering students and professionals.

This book is also supported by my university over the last two years (2015-2016). Finally, I would like to thank my wife, Yi-Teh, for her patience and understanding during the preparation of the book over the last eight years.

Chong-Yung Chi

National Tsing Hua University, Hsinchu, Taiwan December 2016

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1 Mathematical Background

Convex (CVX) optimization is an important class of optimization techniques that includes least squares and linear programs as special cases, and has been extensively used in various science and engineering areas. If one can formulate a practical problem as a convex optimization problem, then actually he (she) has solved the original problem (for an optimal solution either analytically or numerically), like least squares (LS) or linear program, (almost) technology. This chapter provides some essential mathematical basics of vector spaces, norms, sets, functions, matrices, and linear algebra, etc., in order to smoothly introduce the CVX optimization theory from fundamentals to applications in each of the following chapters. It is expected that the CVX optimization theory will be more straightforward and readily understood and learned.

1.1 Mathematical prerequisites

In this section, let us introduce all the notations and abbreviations and some mathematical preliminaries that will be used in the remainder of the book. Our notations and abbreviations are standard, following those widely used in convex optimization for signal processing and communications, that are defined, respectively, as follows:

Notations:

$\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{R}^{m\times n}$	Set of real numbers, n-vectors, $m \times n$ matrices
$\mathbb{C}$, $\mathbb{C}^n$, $\mathbb{C}^{m\times n}$	Set of complex numbers, n-vectors, $m \times n$ matrices
$\mathbb{R}_+$, $\mathbb{R}^n_+$, $\mathbb{R}^{m\times n}_+$	Set of nonnegative real numbers, n-vectors, $m \times n$ matrices
$\mathbb{R}_{++}$, $\mathbb{R}^n_{++}$, $\mathbb{R}^{m\times n}_{++}$	Set of positive real numbers, n-vectors, $m \times n$ matrices
$\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{Z}_{++}$	Set of integers, nonnegative integers, positive integers

$\mathbb{S}^n$, $\mathbb{S}^n_+$, $\mathbb{S}^n_{++}$	Set of $n \times n$ real symmetric matrices, positive semidefinite matrices, positive definite matrices
$\mathbb{H}^n$, $\mathbb{H}^n_+$, $\mathbb{H}^n_{++}$	Set of $n \times n$ Hermitian matrices, positive semidefinite matrices, positive definite matrices
$\{x_i\}_{i=1}^N$	The set $\{x_1, \ldots, x_N\}$
$\mathbf{x} = [x_1, \ldots, x_n]^T = (x_1, \ldots, x_n)$	n-dimensional column vector $\mathbf{x}$
$[\mathbf{x}]_i$	ith component of a vector $\mathbf{x}$
$[\mathbf{x}]_{i:j}$	A column vector constituted by partial elements of the vector $\mathbf{x}$, containing $[\mathbf{x}]_i, [\mathbf{x}]_{i+1}, \ldots, [\mathbf{x}]_j$
$\text{card}(\mathbf{x})$	Cardinality (number of nonzero elements) of a vector $\mathbf{x}$
$\mathbf{Diag}(\mathbf{x})$	Diagonal (square) matrix whose ith diagonal element is the ith element of a vector $\mathbf{x}$
$\mathbf{X} = \{x_{ij}\}_{M \times N} = \{[\mathbf{X}]_{ij}\}_{M \times N}$	$M \times N$ matrix $\mathbf{X}$ with the (i,j)th component $[\mathbf{X}]_{ij} = x_{ij}$
$\mathbf{X}^*$	Complex conjugate of a matrix $\mathbf{X}$
$\mathbf{X}^T$	Transpose of a matrix $\mathbf{X}$
$\mathbf{X}^H = (\mathbf{X}^*)^T$	Hermitian (i.e., conjugate transpose) of a matrix $\mathbf{X}$
$\text{Re}\{\cdot\}$	Real part of the argument
$\text{Im}\{\cdot\}$	Imaginary part of the argument
$\mathbf{X}^\dagger$	Pseudo-inverse of a matrix $\mathbf{X}$
$\text{Tr}(\mathbf{X})$	Trace of a square matrix $\mathbf{X}$
$\mathbf{vec}(\mathbf{X})$	Column vector formed by sequentially stacking all the columns of a square matrix $\mathbf{X}$
$\mathbf{vecdiag}(\mathbf{X})$	Column vector whose elements are the diagonal elements of a square matrix $\mathbf{X}$
$\mathbf{DIAG}(\mathbf{X}_1, \ldots, \mathbf{X}_n)$	Block-diagonal matrix (not necessarily a square matrix), with $\mathbf{X}_1, \ldots, \mathbf{X}_n$ as its diagonal blocks, where $\mathbf{X}_1, \ldots, \mathbf{X}_n$ may not be square matrices
$\text{rank}(\mathbf{X})$	Rank of a matrix $\mathbf{X}$
$\det(\mathbf{X})$	Determinant of a square matrix $\mathbf{X}$
$\lambda_i(\mathbf{X})$	The ith eigenvalue (or ith principal eigenvalue if specified) of a real symmetric (or *Hermitian*) matrix $\mathbf{X}$
$\mathcal{R}(\mathbf{X})$	Range space of a matrix $\mathbf{X}$
$\mathcal{N}(\mathbf{X})$	Null space of a matrix $\mathbf{X}$
$\dim(V)$	Dimension of a subspace V
$\|\cdot\|$	Norm
$\text{span}[\mathbf{v}_1, \ldots, \mathbf{v}_n]$	Subspace spanned by vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$
$\mathbf{1}_n$	All-one column vector of dimension n