PRINCIPLES OF ALGEBRAIC GEOMETRY

PHILLIP GRIFFITHS and JOSEPH HARRIS

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PREFACE

Algebraic geometry is among the oldest and most highly developed subjects in mathematics. It is intimately connected with projective geometry, complex analysis, topology, number theory, and many other areas of current mathematical activity. Moreover, in recent years algebraic geometry has undergone vast changes in style and language. For these reasons there has arisen about the subject a reputation of inaccessibility. This book gives a presentation of some of the main general results of the theory accompanied by—and indeed with special emphasis on—the applications to the study of interesting examples and the development of computational tools.

A number of principles guided the preparation of the book. One was to develop only that general machinery necessary to study the concrete geometric questions and special classes of algebraic varieties around which the presentation was centered.

A second was that there should be an alternation between the general theory and study of examples, as illustrated by the table of contents. The subject of algebraic geometry is especially notable for the balance provided on the one hand by the intricacy of its examples and on the other by the symmetry of its general patterns; we have tried to reflect this relationship in our choice of topics and order of presentation.

A third general principle was that this volume should be self-contained. In particular any "hard" result that would be utilized should be fully proved. A difficulty a student often faces in a subject as diverse as algebraic geometry is the profusion of cross-references, and this is one reason for attempting to be self-contained. Similarly, we have attempted to avoid allusions to, or statements without proofs of, related results. This book is in no way meant to be a survey of algebraic geometry, but rather is designed to develop a working facility with specific geometric questions. Our approach to the subject is initially analytic: Chapters 0 and 1 treat the basic techniques and results of complex manifold theory, with some emphasis on results applicable to projective varieties. Beginning in Chapter 2 with the theory of Riemann surfaces and algebraic curves, and continuing in Chapters 4 and 6 on algebraic surfaces and the quadric line complex, our treatment becomes increasingly geometric along classical lines. Chapters 3 and 5 continue the analytic approach, progressing to more special topics in complex manifolds.

Several important topics have been entirely omitted. The most glaring are the arithmetic theory of algebraic varieties, moduli questions, and singularities. In these cases the necessary techniques are not fully developed here. Other topics, such as uniformization and automorphic forms or monodromy and mixed Hodge structures have been omitted, although the necessary techniques are for the most part available.

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FOUNDATIONAL MATERIAL

In this chapter we sketch the foundational material from several complex variables, complex manifold theory, topology, and differential geometry that will be used in our study of algebraic geometry. While our treatment is for the most part self-contained, it is tacitly assumed that the reader has some familiarity with the basic objects discussed. The primary purpose of this chapter is to establish our viewpoint and to present those results needed in the form in which they will be used later on. There are, broadly speaking, four main points:

1. *The Weierstrass theorems and corollaries*, discussed in Sections 1 and 2. These give us our basic picture of the local character of analytic varieties. The theorems themselves will not be quoted directly later, but the picture—for example, the local representation of an analytic variety as a branched covering of a polydisc—is fundamental. The foundations of local analytic geometry are further discussed in Chapter 5.

2. *Sheaf theory*, discussed in Section 3, is an important tool for relating the analytic, topological, and geometric aspects of an algebraic variety. A good example is the *exponential sheaf sequence*, whose individual terms $\mathbb{Z}$, $\mathcal{O}$, and $\mathcal{O}^*$ reflect the topological, analytic, and geometric structures of the underlying variety, respectively.

3. *Intersection theory*, discussed in Section 4, is a cornerstone of classical algebraic geometry. It allows us to treat the incidence properties of algebraic varieties, a priori a geometric question, in topological terms.

4. *Hodge theory*, discussed in Sections 6 and 7. By far the most sophisticated technique introduced in this chapter, Hodge theory has, in the present context, two principal applications: first, it gives us the *Hodge decomposition* of the cohomology of a Kähler manifold; then, together with the formalism introduced in Section 5, it gives the vanishing theorems of the next chapter.

1. RUDIMENTS OF SEVERAL COMPLEX VARIABLES

Cauchy's Formula and Applications

NOTATION. We will write $z=(z_1,\dots,z_n)$ for a point in $\mathbb{C}^n$, with

$$z_i = x_i + \sqrt{-1}\, y_i;$$
$$\|z\|^2 = (z,z) = \sum_{i=1}^{n} |z_i|^2.$$

For U an open set in $\mathbb{C}^n$, write $C^\infty(U)$ for the set of C^∞ functions defined on U; $C^\infty(\overline{U})$ for the set of C^∞ functions defined in some neighborhood of the closure $\overline{U}$ of U.

The cotangent space to a point in $\mathbb{C}^n \cong \mathbb{R}^{2n}$ is spanned by $\{dx_i, dy_i\}$; it will often be more convenient, however, to work with the complex basis

$$dz_i = dx_i + \sqrt{-1}\, dy_i, \qquad d\bar{z}_i = dx_i - \sqrt{-1}\, dy_i$$

and the dual basis in the tangent space

$$\frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i}\right), \qquad \frac{\partial}{\partial \bar{z}_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} + \sqrt{-1}\frac{\partial}{\partial y_i}\right).$$

With this notation, the formula for the total differential is

$$df = \sum_i \frac{\partial f}{\partial z_i}\, dz_i + \sum_j \frac{\partial f}{\partial \bar{z}_j}\, d\bar{z}_j.$$

In one variable, we say a C^∞ function f on an open set $U \subset \mathbb{C}$ is *holomorphic* if f satisfies the Cauchy-Riemann equations $\partial f/\partial \bar{z} = 0$. Writing $f(z) = u(z) + \sqrt{-1}\, v(z)$, this amounts to

$$\mathrm{Re}\left(\frac{\partial f}{\partial \bar{z}}\right) = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0,$$
$$\mathrm{Im}\left(\frac{\partial f}{\partial \bar{z}}\right) = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0.$$

We say f is *analytic* if, for all $z_0 \in U$, f has a local series expansion in $z - z_0$, i.e.,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

in some disc $\Delta(z_0, \varepsilon) = \{z : |z - z_0| < \varepsilon\}$, where the sum converges absolutely and uniformly. The first result is that f is analytic if and only if it is holomorphic; to show this, we use the

Cauchy Integral Formula. *For Δ a disc in $\mathbb{C}$, $f \in C^\infty(\overline{\Delta})$, $z \in \Delta$,*

$$f(z) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial\Delta} \frac{f(w)\, dw}{w - z} + \frac{1}{2\pi\sqrt{-1}} \int_{\Delta} \frac{\partial f(w)}{\partial \bar{w}} \frac{dw \wedge d\bar{w}}{w - z},$$

where the line integrals are taken in the counterclockwise direction (the fact that the last integral is defined will come out in the proof).

Proof. The proof is based on Stokes' formula for a differential form with singularities, a method which will be formalized in Chapter 3. Consider the differential form

$$\eta = \frac{1}{2\pi\sqrt{-1}} \frac{f(w)\,dw}{w-z};$$

we have for $z \neq w$

$$\frac{\partial}{\partial \bar{w}}\left(\frac{1}{w-z}\right) = 0$$

and so

$$d\eta = -\frac{1}{2\pi\sqrt{-1}} \frac{\partial f(w)}{\partial \bar{w}} \frac{dw \wedge d\bar{w}}{w-z}.$$

Let $\Delta_\varepsilon = \Delta(z,\varepsilon)$ be the disc of radius ε around z. The form η is C^∞ in $\Delta - \Delta_\varepsilon$, and applying Stokes' theorem we obtain

$$\frac{1}{2\pi\sqrt{-1}} \int_{\partial\Delta_\varepsilon} \frac{f(w)\,dw}{w-z} = \frac{1}{2\pi\sqrt{-1}} \int_{\partial\Delta} \frac{f(w)\,dw}{w-z} + \frac{1}{2\pi\sqrt{-1}} \int_{\Delta-\Delta_\varepsilon} \frac{\partial f}{\partial \bar{w}} \frac{dw \wedge d\bar{w}}{w-z}.$$

Setting $w - z = re^{i\theta}$,

$$\frac{1}{2\pi\sqrt{-1}} \int_{\partial\Delta_\varepsilon} \frac{f(w)\,dw}{w-z} = \frac{1}{2\pi} \int_0^{2\pi} f(z+\varepsilon e^{i\theta})\,d\theta,$$

which tends to $f(z)$ as $\varepsilon \to 0$; moreover,

$$dw \wedge d\bar{w} = -2\sqrt{-1}\;dx \wedge dy = -2\sqrt{-1}\;r\,dr \wedge d\theta$$

so

$$\left| \frac{\partial f(w)}{\partial \bar{w}} \frac{dw \wedge d\bar{w}}{w-z} \right| = 2\left| \frac{\partial f}{\partial \bar{w}}\, dr \wedge d\theta \right| \leq c\,|dr \wedge d\theta|.$$

Thus $(\partial f/\partial \bar{w})(dw \wedge d\bar{w})/(w-z)$ is absolutely integrable over Δ, and

$$\int_{\Delta_\varepsilon} \frac{\partial f}{\partial \bar{w}} \frac{dw \wedge d\bar{w}}{w-z} \to 0$$

as $\varepsilon \to 0$; the result follows. Q.E.D.

Now we can prove the

Proposition. *For* U *an open set in* $\mathbb{C}$ *and* $f \in C^\infty(U)$, f *is holomorphic if and only if* f *is analytic.*

Proof. Suppose first that $\partial f/\partial\bar{z}=0$. Then for $z_0\in U$, ε sufficiently small, and z in the disc $\Delta=\Delta(z_0,\varepsilon)$ of radius ε around z_0,

$$\begin{aligned}
f(z) &= \frac{1}{2\pi\sqrt{-1}}\int_{\partial\Delta}\frac{f(w)\,dw}{w-z}\\
&= \frac{1}{2\pi\sqrt{-1}}\int_{\partial\Delta}\frac{f(w)\,dw}{(w-z_0)-(z-z_0)}\\
&= \frac{1}{2\pi\sqrt{-1}}\int_{\partial\Delta}\frac{f(w)\,dw}{(w-z_0)\left(1-\dfrac{z-z_0}{w-z_0}\right)}\\
&= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi\sqrt{-1}}\int_{\partial\Delta}\frac{f(w)\,dw}{(w-z_0)^{n+1}}\right)(z-z_0)^n;
\end{aligned}$$

so, setting

$$a_n=\frac{1}{2\pi\sqrt{-1}}\int_{\partial\Delta}\frac{f(w)\,dw}{(w-z_0)^{n+1}},$$

we have

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n$$

for $z\in\Delta$, where the sum converges absolutely and uniformly in any smaller disc.

Suppose conversely that $f(z)$ has a power series expansion

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n$$

for $z\in\Delta=\Delta(z_0,\varepsilon)$. Since $(\partial/\partial\bar{z})(z-z_0)^n=0$, the partial sums of the expansion satisfy Cauchy's formula without the area integral, and by the uniform convergence of the sum in a neighborhood of z_0 the same is true of f, i.e.,

$$f(z)=\frac{1}{2\pi\sqrt{-1}}\int_{\partial\Delta}\frac{f(w)\,dw}{w-z}.$$

We can then differentiate under the integral sign to obtain

$$\frac{\partial}{\partial\bar{z}}f(z)=\frac{1}{2\pi\sqrt{-1}}\int_{\partial\Delta}\frac{\partial}{\partial\bar{z}}\left(\frac{f(w)}{w-z}\right)dw=0,$$

since for $z\neq w$

$$\frac{\partial}{\partial\bar{z}}\left(\frac{1}{w-z}\right)=0.$$

Q.E.D.

We prove a final result in one variable, that given a C^∞ function g on a disc Δ the equation

$$\frac{\partial f}{\partial \bar{z}} = g$$

can always be solved on a slightly smaller disc; this is the

$\bar{\partial}$-Poincaré Lemma in One Variable. *Given* $g(z) \in C^\infty(\bar{\Delta})$, *the function*

$$f(z) = \frac{1}{2\pi\sqrt{-1}} \int_\Delta \frac{g(w)}{w-z} \, dw \wedge d\bar{w}$$

is defined and C^∞ *in* Δ *and satisfies*

$$\frac{\partial f}{\partial \bar{z}} = g.$$

Proof. For $z_0 \in \Delta$ choose ε such that the disc $\Delta(z_0, 2\varepsilon) \subset \Delta$ and write

$$g(z) = g_1(z) + g_2(z),$$

where $g_1(z)$ vanishes outside $\Delta(z_0, 2\varepsilon)$ and $g_2(z)$ vanishes inside $\Delta(z_0, \varepsilon)$. The integral

$$f_2(z) = \frac{1}{2\pi\sqrt{-1}} \int_\Delta g_2(w) \frac{dw \wedge d\bar{w}}{w-z}$$

is well-defined and C^∞ for $z \in \Delta(z_0, \varepsilon)$; there we have

$$\frac{\partial}{\partial \bar{z}} f_2(z) = \frac{1}{2\pi\sqrt{-1}} \int_\Delta \frac{\partial}{\partial \bar{z}} \left(\frac{g_2(w)}{w-z} \right) dw \wedge d\bar{w} = 0.$$

Since $g_1(z)$ has compact support, we can write

$$\begin{aligned} \frac{1}{2\pi\sqrt{-1}} \int_\Delta g_1(w) \frac{dw \wedge d\bar{w}}{w-z} &= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbf{C}} g_1(w) \frac{dw \wedge d\bar{w}}{w-z} \\ &= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbf{C}} g_1(u+z) \frac{du \wedge d\bar{u}}{u}, \end{aligned}$$

where $u = w - z$. Changing to polar coordinates $u = re^{i\theta}$ this integral becomes

$$f_1(z) = -\frac{1}{\pi} \int_{\mathbf{C}} g_1(z + re^{i\theta}) e^{-i\theta} \, dr \wedge d\theta,$$

which is clearly defined and C^∞ in z. Then

$$\begin{aligned} \frac{\partial f_1(z)}{\partial \bar{z}} &= -\frac{1}{\pi} \int_{\mathbf{C}} \frac{\partial g_1}{\partial \bar{z}} (z + re^{i\theta}) e^{-i\theta} \, dr \wedge d\theta \\ &= \frac{1}{2\pi\sqrt{-1}} \int_\Delta \frac{\partial g_1}{\partial \bar{w}}(w) \frac{dw \wedge d\bar{w}}{w-z}; \end{aligned}$$

but g_1 vanishes on $\partial\Delta$, and so by the Cauchy formula

$$\frac{\partial}{\partial \bar{z}} f(z) = \frac{\partial}{\partial \bar{z}} f_1(z) = g_1(z) = g(z). \qquad \text{Q.E.D.}$$

Several Variables

In the formula

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial z_i} dz_i + \sum_{i=1}^{n} \frac{\partial f}{\partial \bar{z}_i} d\bar{z}_i$$

for the total differential of a function f on $\mathbb{C}^n$, we denote the first term ∂f and the second term $\bar{\partial} f$; ∂ and $\bar{\partial}$ are differential operators invariant under a complex linear change of coordinates. A C^∞ function f on an open set $U \subset \mathbb{C}^n$ is called *holomorphic* if $\bar{\partial} f = 0$; this is equivalent to $f(z_1, \ldots, z_n)$ being holomorphic in each variable z_i separately.

As in the one-variable case, a function f is holomorphic if and only if it has local power series expansions in the variables z_i. This is clear in one direction: by the same argument as before, a convergent power series defines a holomorphic function. We check the converse in the case $n=2$; the computation for general n is only notationally more difficult. For f holomorphic in the open set $U \subset \mathbb{C}^2$, $z_0 \in U$, we can fix Δ the disc of radius r around $z_0 \in U$ and apply the one-variable Cauchy formula twice to obtain, for $(z_1, z_2) \in \Delta$,

$$\begin{aligned} f(z_1,z_2) &= \frac{1}{2\pi\sqrt{-1}} \int_{|w_2 - z_{0_2}| = r} \frac{f(z_1, w_2)\, dw_2}{w_2 - z_2} \\ &= \frac{1}{2\pi\sqrt{-1}} \int_{|w_2 - z_{0_2}| = r} \left[\frac{1}{2\pi\sqrt{-1}} \int_{|w_1 - z_{0_1}| = r} \frac{f(w_1, w_2)\, dw_1}{w_1 - z_1} \right] \frac{dw_2}{w_2 - z_2} \\ &= \left(\frac{1}{2\pi\sqrt{-1}} \right)^2 \int\int_{|w_i - z_{0_i}| = r} \frac{f(w_1, w_2)\, dw_1\, dw_2}{(w_1 - z_1)(w_2 - z_2)}. \end{aligned}$$

Using the series expansion

$$\frac{1}{(w_1 - z_1)(w_2 - z_2)} = \sum_{m,n=0}^{\infty} \frac{(z_1 - z_{0_1})^m (z_2 - z_{0_2})^n}{(w_1 - z_{0_1})^{m+1} (w_2 - z_{0_2})^{n+1}},$$

we find that f has a local series expansion

$$f(z_1,z_2) = \sum_{m,n=0}^{\infty} a_{m,n} (z_1 - z_{0_1})^m (z_2 - z_{0_2})^n. \qquad \text{Q.E.D.}$$