Convolution Integral Equations

with special function kernels

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PREFACE

A Volterra integral equation of the first kind can be written in the form

$$(0.1) \qquad \int_a^x k(x,t)\, f(t)\, dt = g(x),$$

where $a > -\infty$. It is known that for certain cases this equation can be converted into a Volterra equation of the second kind by differentiation. It is our purpose to consider alternative methods of solving the equations of the first kind where the kernels are given special functions, since quite often the requirements for conversion to equations of the second kind are not satisfied. In particular, we concentrate on a survey of the kernels and of the methods for which explicit integral (or differential) inversion formulas can be obtained.

In Chapter 1 we give a discussion of those kernels which are available in the literature with references to the lengthy bibliography which has been collected. Chapter 2 consists of a discussion of relations between various forms of the equations, a discussion of the question of uniqueness of solutions, taking into consideration the Theorem of Titchmarsh and the work of Gesztelyi, and a discussion of the related integral transformations and fractional integrals. A description of each of several methods is given in Chapter 3 with an explicit, simple example for the illustration of each method. Further, in Chapter 4 we include some miscellaneous results and mention various problems which may be worthy of further investigation.

Some notations which we use for special functions appear in the Appendix, and, in general, these notations are the ones used in "Higher Transcendental Functions" by A. Erdélyi et al. [41]. We shall refer often to "Tables of Integral Transforms", by A. Erdélyi et al. [42] for integrals. In both of these cases we use abbreviations such as [41:2.1(2)] to refer to the reference numbered 41, chapter 2, section 1, formula (2).

Integral transformations are mentioned as methods of solving integral equations of the first kind in a number of books; for example, R.V. Churchill [27], V.A. Ditkin and A.P. Prudnikov [30], Gustav Doetsch [32,34], Balth. van der Pol and H. Bremmer [80], Ian N. Sneddon [115, 117], and E.C. Titchmarsh [140]; the operational calculus of Mikusiński is used by Lothar Berg [1] and Arthur Erdélyi [37]. For discussions of convolutions and their properties, besides the already listed texts, one can refer to Gustav Doetsch [33] for a lengthy discussion of the simple convolution and to István Fenyö [44] and to E. Gesztelyi [48] for generalizations. J.G. Mikusiński and Cz. Ryll-Nardzewski [72] give a table relating the class to which the convolution product belongs to the classes from which the factors come.

For lack of a brief common terminology to describe the method in which a key integral involving two kernels k and k_1 is computed and then a direct verification of the proposed solution is made by substitution into the original equation, we introduce the expression "resolvent kernel method". Further discussion and an example of this method is contained in section 2 of Chapter 3.

Although we have restricted our discussion mostly to forms related to equation (0.1), we do include a few cases of somewhat related equations of the first kind for which the interval is of the form (x,∞); the cases of other equations of the first kind where the intervals for integration are (a,b) or $(-\infty, +\infty)$ are quite different problems. Since a large variety of physical problems lead to convolution integral equations with special function kernels, it is hoped that the types of solutions explored here will be useful in the disciplines of applied mathematics, theoretical and quantum mechanics, and mathematical physics.

The collaboration on this project was initiated, and a preliminary draft was prepared, during the academic year 1972-73 while the second author was at the University of Victoria on sabbatical leave from the University of Wyoming. This monograph has since been revised and updated a number of times.

Victoria
and
Guelph
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CHAPTER 1

LITERATURE ON SPECIAL FUNCTION KERNELS

1.1 Algebraic kernels

The books mentioned in the introduction as well as many of the papers which will be mentioned in this chapter treat Abel's equation,

$$(1.1.1) \qquad \int_a^x (x-t)^{-\alpha} f(t)\,dt = g(x), \qquad 0 < \alpha < 1,$$

as an example in view of its simplicity and also because it follows as a special case of other kernels. This equation has a long history which we have not specifically pursued. The inversions of certain fractional integrals are definitely related to the problem of the solution of Abel's equation or its generalizations: see, for example, L.S. Bosanquet [10], A. Erdélyi [35, 36], H. Kober [63], J.S. Lowndes [68], and C.V.L. Smith [114]. Both intervals (a, x) and (x, ∞) have been considered in the literature.

Certain other equations can be reduced to Abel's equation. For example, the equation

$$(1.1.2) \qquad \int_a^x \left[k(x) - k(t)\right]^{-\alpha} f(t)\,dt = g(x), \qquad 0 < \alpha < 1,$$

can sometimes be reduced to an Abel equation by a simple change of variables; see J. Burlak [13], A.N. Hovanskii [56], R.P. Srivastav [121], and E. Gesztelyi [48].

The related problem involving the Hadamard finite part of a divergent integral when $\alpha \geq 1$ is considered by T.K. Boehme [8], P.L. Butzer [21, 22], Erdélyi [37], Fritz Rühs [90], and Klaus Wiener [145, 146] and a series of related papers.

For integral transformation and operational calculus techniques for algebraic kernels see also Butzer [20], Ky Fan [43], C. Fox [46], Ram P. Kanwal [61], and Sneddon [116].

Although we are not considering numerical methods here, a recent paper by Richard Weiss [142] indicates references in that direction.

1.2 Exponential, logarithmic, and trigonometric kernels

Some kernels involving the exponential function, for which the technique of Laplace transforms is applied, have been treated by Kanwal [61] and by D.O. Reudink [88]. The general problem of finding inversion integrals which involve the same kernel function was studied by Reudink [loc. cit.]; Fenyö [45] also treated this problem of Reudink by means of Mikusiński operators. Kernels which involve logarithms were discussed by S. Colombo [29], Ky Fan [43], and L. Poli [81], and for Hadamard finite parts by Wiener [145]. Trigonometric and hyperbolic functions as kernels are included in the works of Butzer [20], Erdélyi [37], Kanwal [61], van der Pol and Bremmer [80], Reudink [88], K.C. Rusia [98], and Sneddon [115, 117]. Generalized hyperbolic and trigonometric functions [41:18.2] were used as kernels by P.L. Bharatiya [4].

1.3 Chebyshev polynomials

Much of the recent interest in these equations seems to have been initiated by the papers of Ta Li [64, 65] who solved

$$(1.3.1) \qquad \int_x^1 \left(t^2-x^2\right)^{-1/2} T_n(t/x)\, f(t)\, dt = g(x),$$

in which $T_n(x)$ denotes a Chebyshev polynomial of the first kind of degree n in x. This equation arose from problems in aerodynamics. He used the resolvent kernel method and he needed to evaluate the key integral involving the product of kernels,

$$(1.3.2) \qquad \int_x^v \left(t^2-x^2\right)^{-1/2} T_n(t/x) \left(v^2-t^2\right)^{-1/2} T_{n-1}(t/v)\, dt.$$

In part, the effort of further research has been directed toward finding simpler methods of evaluating such integrals when they are not already tabulated, as well as toward finding simpler methods of discovering solution forms involving other kernels. Other efforts have been made in obtaining alternative methods for solving the equation; Sneddon [116, 117] has used the Mellin transformation and D.V. Widder [144] used the Laplace transformation after making a change of variables. Dietrich Suschowk [138] considered a related equation using a formula for $T_n(x)$, manipulations, and a limiting process.

In the case of the Chebyshev kernels, as well as in a number of other cases which follow, the solution could be obtained by specializing the solution of an integral equation with a more general function as the kernel.

1.4 Legendre polynomials and functions

In a number of papers kernels which involve Legendre polynomials have been considered; for example, the simple equation

$$(1.4.1) \qquad \int_x^1 P_n(t/x)\, f(t)\, dt = g(x),$$

was solved by Buschman [14] using the resolvent kernel method, by Erdélyi [38] using Rodrigues' formula followed by successive integra'ions and differentiations, by Sneddon [116] using the Mellin transformation, and by Widder [144] using the Laplace transformation. A similar equation with a kernel involving generalized Legendre polynomials which are closely related to special cases of Jacobi polynomials was treated by R.P. Singh [112]. The equation

$$(1.4.2) \qquad \int_0^x P_n\big(\cos \alpha(x-t)\big)\, f(t)\, dt = g(x),$$

and its solution appeared in the list of van der Pol and Bremmer [80] for $\alpha = 1$ and is also considered by B.R. Bhonsle [6]. Mikiharu Terada [139] applied Mikusinski operators to a similar equation with kernel $P_n(e^u)$.

A quite different method using partial differential equations was invoked by A.G. Mackie [69] to solve the equations

$$(1.4.3) \qquad \int_0^x P_n(t/x)\, f(t)\, dt = g(x)$$

and

$$(1.4.4) \qquad \int_0^x P_n(x/t)\, f(t)\, dt = g(x).$$

The first of these equations was also treated in the series of three papers, R.A. Sack [100], Ll.C. Chambers and R.A. Sack [25], and Ll.C. Chambers [24]. Equations of this type have quite different properties from those mentioned in the preceding paragraph; the solutions need not be unique because of orthogonality relations. This is discussed in Chapter 2.

For equations with kernels involving Legendre functions; specifically

$$(1.4.5) \qquad \int_x^1 \left(t^2-x^2\right)^{\lambda/2} P_\nu^\lambda(t/x)\, f(t)\, dt = g(x)$$

and

$$(1.4.6) \qquad \int_1^x \left(x^2-t^2\right)^{\lambda/2} \underset{\sim}{P}{}_\nu^\lambda(t/x)\, f(t)\, dt = g(x),$$

with $\operatorname{Re}(\lambda) < 1$, solutions were given by Buschman [16] by means of the resolvent kernel method. Erdélyi [39, 40] used the method of fractional integration to factor the operator into a product of two fractional integrals and hence obtained the solution for the range of integration (a, x); further, in the second paper, he treated the problem of Hadamard finite parts. The Mellin transform was applied by Sneddon [116] to solve the equation for the range (x, ∞) with $x > 0$. Solutions can be obtained as special cases of ${}_2F_1$-kernels such as has been done recently by Tilak Raj Prabhakar [86].

1.5 Gegenbauer and Jacobi polynomials

Solutions to equations of the form

$$(1.5.1) \qquad \int_x^1 \left(t^2-x^2\right)^{\lambda-1/2} C_n^{\lambda}(t/x)\, f(t)\, dt = g(x),$$

which involve Gegenbauer polynomials in the kernel, were given by Buschman [15] for a special case, by Theodore P. Higgins [53], and by K.N. Srivastava [129] (where the corrected results of [127] appear); the resolvent kernel method was used. Sneddon [116] used the Mellin transformation and Prabhakar [86] listed the solution among special cases of the ${}_2F_1$-kernel.

In an earlier paper K.N. Srivastava [126] applied the Hankel transformation to obtain solutions to similar equations where the ranges of integration are $(0, x)$ and (x, ∞).

Equations involving the Jacobi polynomials in the form

$$(1.5.2) \qquad \int_x^1 \left(t^2-x^2\right)^{\alpha} P_n^{(\alpha,\beta)}\left(2t^2/x^2-1\right) f(t)\, dt = g(x),$$

have been studied by K.C. Rusia [91] and K.N. Srivastava [128, 130, 132, 133], both using resolvent kernels. On the other hand, Rusia [95] used Rodrigues' formula for the case in which α is a non-negative integer, and in [97] he applied the Hankel transformation. Prabhakar [86] listed the equation among the special cases of ${}_2F_1$-kernels. C. Singh [110] used the fractional derivative form of Rodrigues' formula to solve the equations with the kernels involving $P_n^{(\alpha,\beta)}(1-2x/t)$; Bhonsle [7] considered kernels involving $P_n^{(\alpha,\beta)}\left(1-2x^2/t^2\right)$.

Jacobi functions occurred in the kernels of K.N. Srivastava [134] where the ranges of integration (0, x) and (x, ∞) were considered and the Hankel transformation applied.

1.6 Laguerre and Hermite polynomials

In these cases the equation is taken in the form of a Laplace convolution. For the Laguerre polynomial kernels we have the equation

$$(1.6.1) \qquad \int_0^x e^{b(x-t)}(x-t)^{\alpha} L_n^{(\alpha)}(x-t)\, f(t)\, dt = g(x),$$

which was treated by Rusia [92]. For the case in which α is a non-negative integer Rusia [95] used Rodrigues' formula. For the case $b = 0$ Buschman [17] used Mikusiński operators, P.R. Khandekar [62] used the Laplace transformation, and K.N. Srivastava [135] used the resolvent kernel method. For $\alpha = b = 0$ Hisachi Choda and Marie Echigo [26] used Mikusiński operators and van der Pol and Bremmer [80] and Widder [143] used the Laplace transformation.

M.T. Shah [114] applied the resolvent kernel method and Laplace transforms to the equation

$$(1.6.2) \qquad \int_0^x (x-t)^{-1/2} H_{2n}\left(\alpha(x-t)^{1/2}\right) f(t)\, dt = g(x),$$

which involves Hermite polynomials. Notice that since

$$(1.6.3) \qquad H_{2n}(x) = (-1)^n\, 2^{2n}\, n!\, L_n^{(-1/2)}(x^2),$$

Shah's solution of (1.6.2) is evidently contained in that of the well-treated equation (1.6.1) with $\alpha = -1/2$ and $b = 0$.

Prabhakar [82] obtained Laguerre and Hermite kernels as special cases of his results on confluent hypergeometric functions.