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（影印版）20

W. Hackbusch

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Theory and Numerical Treatment

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理论与数值处理

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《国外数学名著系列》(影印版)序

要使我国的数学事业更好地发展起来，需要数学家淡泊名利并付出更艰苦地努力。另一方面，我们也要从客观上为数学家创造更有利的发展数学事业的外部环境，这主要是加强对数学事业的支持与投资力度，使数学家有较好的工作与生活条件，其中也包括改善与加强数学的出版工作。

从出版方面来讲，除了较好较快地出版我们自己的成果外，引进国外的先进出版物无疑也是十分重要与必不可少的。从数学来说，施普林格（Springer）出版社至今仍然是世界上最具权威的出版社。科学出版社影印一批他们出版的好的新书，使我国广大数学家能以较低的价格购买，特别是在边远地区工作的数学家能普遍见到这些书，无疑是对推动我国数学的科研与教学十分有益的事。

这次科学出版社购买了版权，一次影印了 23 本施普林格出版社出版的数学书，就是一件好事，也是值得继续做下去的事情。大体上分一下，这 23 本书中，包括基础数学书 5 本，应用数学书 6 本与计算数学书 12 本，其中有些书也具有交叉性质。这些书都是很新的，2000 年以后出版的占绝大部分，共计 16 本，其余的也是 1990 年以后出版的。这些书可以使读者较快地了解数学某方面的前沿，例如基础数学中的数论、代数与拓扑三本，都是由该领域大数学家编著的“数学百科全书”的分册。对从事这方面研究的数学家了解该领域的前沿与全貌很有帮助。按照学科的特点，基础数学类的书以“经典”为主，应用和计算数学类的书以“前沿”为主。这些书的作者多数是国际知名的大数学家，例如《拓扑学》一书的作者诺维科夫是俄罗斯科学院的院士，曾获“菲尔兹奖”和“沃尔夫数学奖”。这些大数学家的著作无疑将会对我国的科研人员起到非常好的指导作用。

当然，23 本书只能涵盖数学的一部分，所以，这项工作还应该继续做下去。更进一步，有些读者面较广的好书还应该翻译成中文出版，使之有更大的读者群。

总之，我对科学出版社影印施普林格出版社的部分数学著作这一举措表示热烈的支持，并盼望这一工作取得更大的成绩。

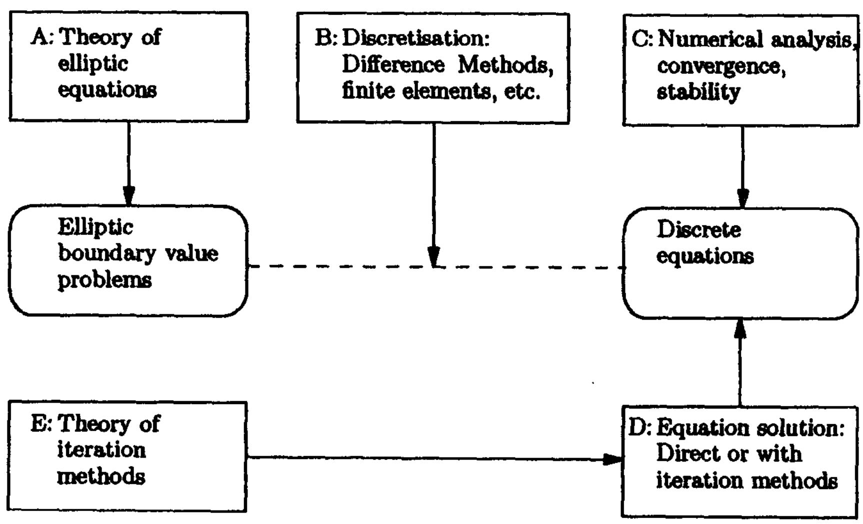
王　元

2005 年 12 月 3 日

Foreword

This book has developed from lectures that the author gave for mathematics students at the Ruhr-Universität Bochum and the Christian-Albrechts-Universität Kiel. This edition is the result of the translation and correction of the German edition entitled *Theorie und Numerik elliptischer Differentialgleichungen*.

The present work is restricted to the theory of partial differential equations of elliptic type, which otherwise tends to be given a treatment which is either too superficial or too extensive. The following sketch shows what the problems are for elliptic differential equations.



The theory of elliptic differential equations (A) is concerned with questions of existence, uniqueness, and properties of solutions. The first problem of

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numerical treatment is the description of the discretisation procedures (B), which give finite-dimensional equations for approximations to the solutions. The subsequent second part of the numerical treatment is numerical analysis (C) of the procedure in question. In particular it is necessary to find out if, and how fast, the approximation converges to the exact solution. The solution of the finite-dimensional equations (D, E) is in general no simple problem, since from 10^3 to 10^6 unknowns can occur. The discussion of this third area of numerical problems is skipped (one may find it, e.g., in Hackbusch [5] and [9]).

The descriptions of discretisation procedures and their analyses are closely connected with corresponding chapters of the theory of elliptic equations. In addition, it is not possible to undertake a well-founded numerical analysis without a basic knowledge of elliptic differential equations. Since the latter cannot, in general, be assumed of a reader, it seems to me necessary to present the numerical study along with the theory of elliptic equations.

The book is conceived in the first place as an introduction to the treatment of elliptic boundary-value problems. It should, however, serve to lead the reader to further literature on special topics and and applications. It is intentional that certain topics, which are often handled rather summarily, (e.g., eigenvalue problems) are treated here in greater detail.

The exposition is strictly limited to linear elliptic equations. Thus a discussion of the Navier-Stokes equations, which are important for fluid mechanics, is excluded; however, one can approach these matters via the Stokes equation, which is thoroughly treated as an example of an elliptic system.

In order not to exceed the limits of this book, we have not considered further discretisation methods (collocation methods, volume-element methods, spectral methods) and integral-equation methods (boundary-element methods).

The Exercises that are presented, which may be considered as remarks without proofs, are an integral part of the exposition. If this book is used as the text for a course they can be used as student problems. But the reader too should test his understanding of the subject on the exercises.

The author wishes to thank his collaborators G. Hofmann, G. Wittum and J. Burmeister for the help in reading and correcting the manuscript of this book. He thanks Teubner Verlag for their cordial collaboration in producing the first German edition.

Kiel, December 1985 W. Hackbusch

This translation contains, in addition to the full text of the original edition, a short Section (§3.5) on the integral-equation method. The bibliography has also been expanded.

The author wishes to thank the translators, R. Fadiman and P. D. F. Ion, for their pleasant collaboration, and Springer-Verlag for their friendly cooperation.

Kiel, March 1992 W. Hackbusch

Notation

Formula Numbers.

Equations in Section $X.Y$ are numbered $(X.Y.1)$, $(X.Y.2)$, etc. The equation (3.2.1) is referred to within Section 3.2 simply as (1). In other Sections of Chapter 3 it is called (2.1).

Theorem Numbering.

All Theorems, Definitions, Lemmata etc. are numbered together. In Section 3.2 Lemma 3.2.1 is referred to as Lemma 1.

Special Symbols. The following quantities have fixed meanings:

$\mathbf{A}, \mathbf{B}, \ldots$	matrices
B, B_j	boundary differential operators (cf. (5.2.1a,b), (5.3.6))
$\mathbb{C}$	the complex numbers
$C^s(D), C^{k,1}(D)$	Hölder- and Lipschitz-continuously differentiable functions (cf. Definition 3.2.8)
$C^k(D), C^\infty(D)$	k-fold and infinitely continuously differentiable functions
$C_0^\infty(\Omega)$	infinitely differentiable functions with compact supports (cf. (6.2.3))
$d(u, V_N)$	distance of the function u from the subspace V_N (cf. Theorem 8.2.1)
$d\Gamma, d\Gamma_{\mathbf{x}}$	surface differentials in surface integrals
$\operatorname{diag}\{d_1, d_2, \ldots\}$	diagonal matrix with the diagonal elements $d_1, d_2, \ldots$
f	a function; often the right-hand side of a differential equation
$g(\cdot,\cdot)$	Green's function (cf. Section 3.2)
h	step size (cf. Sections 4.1 and 4.2)
$H^k(\Omega), H^s(\Omega), H_0^k(\Omega), H_0^s(\Omega)$	Sobolev spaces (cf. Sections 6.2.2 and 6.2.4)
$K_R(y)$	open ball about y with radius R (cf. (2.2.7), Section 6.1.1)
I	identity or inclusion (cf. Sections 6.1.2, 6.1.3)
L	a differential operator (cf. (1.2.6)) or the operator associated with a bilinear form (cf. (7.2.9′))
$\mathbf{L}$	the stiffness matrix (cf. Section 8.1)
L_h	matrix of a discrete system of equations (cf. (4.1.9a))

$L(X,Y)$	linear space of bounded operators from X to Y (cf. Section 6.1.2)
$L^\infty(\Omega)$	space of essentially bounded functions (cf. 6.1.3)
$L^2(\Omega)$	space of square-integrable functions (cf. Section 6.2.1)
$\mathbb{N}$	the natural numbers $\{1,2,3,\ldots\}$
$\mathbf{n} = \mathbf{n}(\mathbf{x})$	normals (cf. (2.2.3a))
$\mathbf{0}$	the zero matrix
$O(\cdot)$	Landau symbol: $f(x) = O(g(x))$ if $\lvert fx)\rvert \le \text{const}\,\lvert g(x)\rvert$
$\mathbf{P}$	cf. (8.1.6)
q_h	a grid function, right-hand side of the discrete equation (4.1.9a)
$\mathbb{R}, \mathbb{R}_+$	the real numbers, the positive real numbers
$R_h, \tilde{R}_h$	restrictions (cf. (4.5.2) and (4.5.5b))
$s(\cdot,\cdot)$	the singularity function (cf. Section 2.2)
$\operatorname{supp}(f)$	the support of the function f (cf. Lemma 6.2.2)
$u = u(\mathbf{x}) = u(x_1,\ldots,x_n)$	a function, e.g., a solution of a differential equation
u_h	a grid function (discrete solution; cf Section 4)
V_N, V_h	finite-element spaces (cf. (8.1.3) and Section 8.4.1)
$x, (x,y), (x,y,z)$	independent variables
$\mathbf{x} = (x_1,\ldots,x_n)$	a vector of independent variables
$\mathbb{Z}$	the integers
Γ	the boundary of Ω
Γ_h	the boundary points of a grid (cf. (4.2.1b), (4.8.4))
$\gamma(\cdot,\cdot)$	fundamental solution function
$\rho(\mathbf{A})$	spectral radius of the matrix $\mathbf{A}$
Ω	an open set in $\mathbb{R}^n$ or a domain (cf. Definition 2.1.1)
Ω_h	a grid (cf. (4.1.6a), (4.2.1a), (4.8.2))
Δ	the Laplace operator (cf. (2.1.1a))
Δ_h	the five-point difference operator (cf. (4.2.3))
∇	gradient (cf. (2.2.3a))
$\partial^+, \partial^-, \partial^0, \partial_x^+, \partial_y^+, \ldots$	differences (cf. (4.1.2a–c), (4.2.3))
$\partial_n^+, \partial_n^-$	differences in the normal direction (cf. (4.7.4))
$\partial/\partial n$	normal derivative (cf. (2.2.3a))
$\langle\cdot,\cdot\rangle$	scalar product (cf. (2.2.3c), (4.3.14a))
$\langle\cdot,\cdot\rangle_{X\times X'}$	duality form (cf. Section 6.1.3)
$(\cdot,\cdot)$	scalar product (cf. Section 6.1.4)
$(\cdot,\cdot)_0, (\cdot,\cdot)_{L^2(\Omega)}, \lvert\cdot\rvert_0, \lvert\cdot\rvert_{L^2(\Omega)}$	scalar product and norms on $L^2(\Omega)$
$(\cdot,\cdot)_k, (\cdot,\cdot)_s, \lvert\cdot\rvert_k, \lvert\cdot\rvert_s$	scalar products and norms on $H^k(\Omega)$, resp. $H^s(\Omega)$
$\lvert\cdot\rvert$	the Euclidean norm (cf. (2.2.2))
$\lVert\cdot\rVert_2$	the Euclidean norm and the spectral norm (cf. Section 4.3)
$\lvert\cdot\rvert_{\hat{k}}, \lvert\cdot\rvert_{\hat{s}}$	norms equivalent to $\lvert\cdot\rvert_k, \lvert\cdot\rvert_s$ (cf. (6.2.15), (6.2.16b))
$\lVert\cdot\rVert_\infty$	maximum norm (cf. (4.3.3)), row sum norm (4.3.11), or supremum norm (2.4.1), cf. also Section 6.1.1
$\mathbb{1}$	the vector $(1,1,\ldots)$ (cf. Section 4.3)

Table of Contents

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1 Partial Differential Equations and Their Classification Into Types

1.1 Examples

An ordinary differential equation describes a function which depends on only one variable. Unfortunately, for many problems it is not possible to restrict attention to a single variable. Almost all physical quantities depend on the spatial variables x, y , and z and on time t. The time dependence might be omitted for stationary processes, and one might perhaps save one spatial dimension by special geometric assumptions, but even then there would still remain at least two independent variables. Equations that contain the first partial derivatives

$$u_{x_i} = u_{x_i}(x_1, x_2, \cdots, x_n) = \partial u(x_1, \cdots, x_n)/\partial x_i$$

where $(1 \leq i \leq n)$, or even higher partial derivatives $u_{x_i x_j}$, etc., are called partial differential equations.

Unlike ordinary differential equations, partial differential equations cannot be analysed all together. Rather, one distinguishes between three types of equations which have different properties and also require different numerical methods.

Before the characteristics for the types are defined, let us introduce some examples of partial differential equations.

All of the following examples will contain only two independent variables x, y. The first two examples are partial differential equations of first order, since only first partial derivatives occur.

Example 1.1.1. Find a solution $u(x, y)$ of

$$u_y(x, y) = 0. \tag{1.1.1}$$

It is obvious that $u(x, y)$ must be independent of y, i.e., the solution has the form $u(x, y) = \varphi(x)$. Thus $u(x, y) = \varphi(x)$ for some arbitrary φ is a solution of (1).

Equation (1) is a special case of

Example 1.1.2. Find a solution $u(x, y)$ of

$$cu_x - u_y = 0 \qquad (c \text{ constant}). \tag{1.1.2}$$

Let u be a solution. Introduce new coordinates $\xi = x + cy$, $\eta = y$ and define $v(\xi,\eta) := u(x(\xi,\eta), y(\xi,\eta))$ with the aid of $x(\xi,\eta) = \xi - c\eta$, $y(\xi,\eta) = \eta$. Since $v_\eta = u_x x_\eta + u_y y_\eta$ (chain rule) and $x_\eta = -c$, $y_\eta = 1$, it follows from (2) that $v_\eta(\xi,\eta) = 0$. This equation is analogous to (1), and Example 1 shows that $v(\xi,\eta) = \varphi(\xi)$. If one now replaces ξ, η by x, y one obtains

$$u(x,y) = \varphi(x + cy). \tag{1.1.3}$$

Conversely, through (3) one obviously obtains a solution of Equation (2) as long as φ is continuously differentiable.

In order to determine uniquely the solution of an ordinary differential equation $u' - f(u) = 0$ one needs an initial value $u(x_0) = u_0$. The partial differential equation (2) can be augmented by the initial-value function

$$u(x, y_0) = u_0(x) \quad \text{for} \quad x \in \mathbb{R} \tag{1.1.4}$$

on the line $y = y_0$, with y_0 a constant. The comparison of Equations (3) and (4) shows that $\varphi(x+cy_0) = u_0(x)$. Thus φ is determined by $\varphi(x) = u_0(x - cy_0)$. The unique solution of the initial value problem (2) and (4) reads

$$u(x,y) = u_0(x - c(y_0 - y)). \tag{1.1.5}$$

The following three examples involve differential equations of second order.

Example 1.1.3. (Potential equation or Laplace equation) Let Ω be an open subset of $\mathbb{R}^2$. Find a solution of

$$u_{xx} + u_{yy} = 0 \quad \text{in } \Omega. \tag{1.1.6}$$

If one identifies $(x,y) \in \mathbb{R}^2$ with the complex number $z = x + iy \in \mathbb{C}$, the solutions can be given immediately. The real and imaginary parts of any function $f(z)$ holomorphic in Ω are solutions of Equation (6). Three examples are $\operatorname{Re} z^0 = 1$, $\operatorname{Re} z^2 = x^2 - y^2$ and $\operatorname{Re}\log(z - z_0) = \log\sqrt{(x-x_0)^2 + (y-y_0)^2}$ if $z_0 \notin \Omega$. To determine the solution uniquely one needs the boundary values $u(x,y) = \varphi(x,y)$ for all (x,y) on the boundary $\Gamma = \partial\Omega$ of Ω.

Example 1.1.4. (Wave equation) All solutions of

$$u_{xx} - u_{yy} = 0 \tag{1.1.7}$$

are given by

$$u(x,y) = \varphi(x + y) + \psi(x - y) \tag{1.1.8}$$

where φ and ψ are arbitrary twice continuously differentiable functions. Suitable initial values are, for example,

$$u(x,0) = u_0(x), \quad u_y(x,0) = u_1(x), \quad (x \in \mathbb{R}) \tag{1.1.9}$$

where u_0 and u_1 are given functions. If one inserts (8) into (9), one finds $u_0 = \varphi + \psi$, $u_1 = \varphi' - \psi'$, where φ' is the derivative of φ, and infers that

$$\varphi' = (u_1 + u_0')/2, \quad \psi' = (u_0' - u_1)/2.$$

From this one can determine φ and ψ up to constants of integration. One constant can be chosen arbitrarily, for example, by $\varphi(0) = 0$, and the other is determined by $u(0,0) = u_0(0) = \varphi(0) + \psi(0)$.

Exercise 1.1.5. Prove that every solution of the wave equation (7) has the form (8). Hint: Use $\xi = x + y$ and $\eta = x - y$ as new variables.

Example 1.1.6. (Heat equation) Find the solution of

$$u_{xx} - u_y = 0 \tag{1.1.10}$$

(physical interpretation: u = temperature, y = time). The separation of variables $u(x,y) = v(x)w(y)$ gives that for every $c \in \mathbb{R}$

$$u(x,y) = \sin(cx) \cdot \exp(-c^2 y). \tag{1.1.11a}$$

Another solution of (10) for $y > 0$ is

$$u(x,y) = \frac{1}{\sqrt{4\pi y}} \int_{-\infty}^{\infty} u_0(\xi) \exp(-(x-\xi)^2/(4y))\, d\xi, \tag{1.1.11b}$$

where $u_0(\cdot)$ is an arbitrary continuous and bounded function. The initial condition matching Equation (10), in contrast to (9), contains only one function:

$$u(x,0) = u_0(x). \tag{1.1.12}$$

The solution (11b), which initially is defined only for $y > 0$, can be extended continuously to $y = 0$ and there satisfies the initial value requirement (12).

Exercise 1.1.7. Let u_0 be bounded in $\mathbb{R}$ and continuous at x. Then prove that the right side of Equation (11b) converges to $u_0(x)$ for $y \to 0$. Hint: First show that $u(x,y) = u_0(x) + \int_{-\infty}^{+\infty} [u_0(\xi) - u_0(x)] \exp(-(x-\xi)^2/(4y))\, d\xi / \sqrt{4\pi y}$ and then decompose the integral into subintegrals over $[x-\delta, x+\delta]$ and $(-\infty, x-\delta) \cup (x+\delta, \infty)$.

As with ordinary differential equations, equations of higher order can be described by systems of first-order equations. In the following we give some examples.

Example 1.1.8. Let the pair (u, v) be the solution of the system

$$u_x + v_y = 0, \quad v_x + u_y = 0. \tag{1.1.13}$$

If u and v are twice differentiable, the differentiation of (13) yields the equations $u_{xx} + v_{xy} = 0$, $v_{xy} + u_{yy} = 0$, which together imply that $u_{xx} - u_{yy} = 0$. Thus u is a solution of the wave equation (7). The same can be shown for v.

Example 1.1.9. (Cauchy-Riemann differential equations) If u and v satisfy the system

$$u_x + v_y = 0, \quad v_x - u_y = 0 \quad \text{in} \quad \Omega \subset \mathbb{R}^2, \tag{1.1.14}$$

then the same consideration as in Example 8 yields that both u and v satisfy the potential equation (6).

Example 1.1.10. If u and v satisfy the system

$$u_x + v_y = 0, \quad v_x + u = 0, \tag{1.1.15}$$

then v solves the heat equation (10).

A second-order system of interest in fluid mechanics can be found in

Example 1.1.11. (Stokes equations) In the system

$$u_{xx} + u_{yy} - w_x = 0, \tag{1.1.16a}$$

$$v_{xx} + v_{yy} - w_y = 0, \tag{1.1.16b}$$

$$u_x + v_y \qquad = 0 \tag{1.1.16c}$$

u and v denote the flow velocities in x- and y-directions, while w denotes the pressure.

1.2 Classification of Second-Order Equations into Types

The general linear differential equation of second order in two variables reads

$$\begin{aligned} a(x,y)u_{xx} &+ 2b(x,y)u_{xy} + c(x,y)u_{yy} \\ &+ d(x,y)u_x + e(x,y)u_y + f(x,y)u + g(x,y) = 0. \end{aligned} \tag{1.2.1}$$

Definition 1.2.1. (a) Equation (1) is said to be elliptic at (x, y) if

$$a(x,y)c(x,y) - b(x,y)^2 > 0. \tag{1.2.2a}$$

(b) Equation (1) is said to be hyperbolic at (x, y), if

$$a(x,y)c(x,y) - b(x,y)^2 < 0. \tag{1.2.2b}$$

(c) Equation (1) is said to be parabolic at (x, y) if

$$ac - b^2 = 0 \quad \text{and} \quad \operatorname{rank} \begin{bmatrix} a & b & d \\ b & c & e \end{bmatrix} = 2 \quad \text{at } (x,y). \tag{1.2.2c}$$

(d) Equation (1) is said to be elliptic (hyperbolic, parabolic) in $\Omega \subset \mathbb{R}^2$ if it is elliptic (hyperbolic, parabolic) at all $(x, y) \in \Omega$.

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