11

Graduate Texts in Mathematics

Douglas S. Bridges

Foundations of Real and Abstract Analysis

实分析和抽象分析基础

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Dedicated to the memory of my parents:
Douglas McDonald Bridges and Allison Hogg

Sweet Analytics, 'tis thou hast ravished me.

FAUSTUS (Marlowe)

The stone which the builders refused is become the head stone of the corner.

PSALM CXVIII, 22.

...from so simple a beginning endless forms most beautiful and most wonderful have been, and are being, evolved.

THE ORIGIN OF SPECIES (Darwin)

Preface

The core of this book, Chapters 3 through 5, presents a course on metric, normed, and Hilbert spaces at the senior/graduate level. The motivation for each of these chapters is the generalisation of a particular attribute of the Euclidean space $\mathbf{R}^n$: in Chapter 3, that attribute is *distance*; in Chapter 4, *length*; and in Chapter 5, *inner product*. In addition to the standard topics that, arguably, should form part of the armoury of any graduate student in mathematics, physics, mathematical economics, theoretical statistics,..., this part of the book contains many results and exercises that are seldom found in texts on analysis at this level. Examples of the latter are Wong's Theorem (3.3.12) showing that the Lebesgue covering property is equivalent to the uniform continuity property, and Motzkin's result (5.2.2) that a nonempty closed subset of Euclidean space has the unique closest point property if and only if it is convex.

The sad reality today is that, perceiving them as one of the harder parts of their mathematical studies, students contrive to avoid analysis courses at almost any cost, in particular that of their own educational and technical deprivation. Many universities have at times capitulated to the negative demand of students for analysis courses and have seriously watered down their expectations of students in that area. As a result, mathematics majors are graduating, sometimes with high honours, with little exposure to anything but a rudimentary course or two on real and complex analysis, often without even an introduction to the Lebesgue integral.

For that reason, and also in order to provide a reference for material that is used in later chapters, I chose to begin this book with a long chapter providing a fast–paced course of real analysis, covering convergence of sequences and series, continuity, differentiability, and (Riemann and Riemann–Stieltjes) integration. The inclusion of that chapter means that the prerequisite for the book is reduced to the usual undergraduate sequence of courses on calculus. (One–variable calculus would suffice, in theory, but a lack of exposure to more advanced calculus courses would indicate a lack of the mathematical maturity that is the hidden prerequisite for most senior/graduate courses.)

Chapter 2 is designed to show that the subject of differentiation does not end with the material taught in calculus courses, and to introduce the Lebesgue integral. Starting with the Vitali Covering Theorem, the chapter develops a theory of *differentiation almost everywhere* that underpins a beautiful approach to the Lebesgue integral due to F. Riesz [39]. One minor disadvantage of Riesz's approach is that, in order to handle multivariate integrals, it requires the theory of set–valued derivatives, a topic sufficiently involved and far from my intended route through elementary analysis that I chose to omit it altogether. The only place where this might be regarded as a serious omission is at the end of the chapter on Hilbert space, where I require classical vector integration to investigate the existence of weak solutions to the Dirichlet Problem in three–dimensional Euclidean space; since that investigation is only outlined, it seemed justifiable to rely only on the reader's presumed acquaintance with elementary vector calculus. Certainly, one–dimensional integration is all that is needed for a sound introduction to the L_p spaces of functional analysis, which appear in Chapter 4.

Chapters 1 and 2 form Part I (Real Analysis) of the book; Part II (Abstract Analysis) comprises the remaining chapters and the appendices. I have already summarised the material covered in Chapters 3 through 5. Chapter 6, the final one, introduces functional analysis, starting with the Hahn–Banach Theorem and the consequent separation theorems. As well as the common elementary applications of the Hahn–Banach Theorem, I have included some deeper ones in duality theory. The chapter ends with the Baire Category Theorem, the Open Mapping Theorem, and their consequences. Here most of the applications are standard, although one or two unusual ones are included as exercises.

The book has a preliminary section dealing with background material needed in the main text, and three appendices. The first appendix describes Bishop's construction of the real number line and the subsequent development of its basic algebraic and order properties; the second deals briefly with axioms of choice and Zorn's Lemma; and the third shows how some of the material in the chapters—in particular, Minkowski's Separation Theorem—can be used in the theory of Pareto optimality and competitive equilibria in mathematical economics. Part of my motivation in writing Appendix C was to indicate that "mathematical economics" is a far deeper subject than is suggested by the undergraduate texts on calculus and linear algebra that are published under that title.

I have tried, wherever possible, to present proofs so that they translate *mutatis mutandis* into their counterparts in a more abstract setting, such as that of a metric space (for results in Chapter 1) or a topological space (for results in Chapter 3). On the other hand, some results first appear as exercises in one context before reappearing as theorems in another: one example of this is the Uniform Continuity Theorem, which first appears as[1] Exercise (1.4.8: 8) in the context of a compact interval of **R**, and which is proved later, as Corollary (3.3.13), in the more general setting of a compact metric space. I hope that this procedure of double exposure will enable students to grasp the material more firmly.

The text covers just over 300 pages, but the book is, in a sense, much larger, since it contains nearly 750 exercises, which can be classified into at least the following, not necessarily exclusive, types:

- applications and extensions of the main propositions and theorems;
- results that fill in gaps in proofs or that prepare for proofs later in the book;
- pointers towards new branches of the subject;
- deep and difficult challenges for the very best students.

The instructor will have a wide choice of exercises to set the students as assignments or test questions. Whichever ones are set, as with the learning of any branch of mathematics it is essential that the student attempt as many exercises as the constraints of time, energy, and ability permit.

It is important for the instructor/student to realise that many of the exercises—especially in Chapters 1 and 2—deal with results, sometimes major ones, that are needed later in the book. Such an exercise may not clearly identify itself when it first appears; if it is not attempted then, it will provide revision and reinforcement of that material when the student needs to tackle it later. It would have been unreasonable of me to have included major results as exercises without some guidelines for the solution of the nonroutine ones; in fact, a significant proportion of the exercises of all types come with some such guideline, even if only a hint.

Although Chapters 3 through 6 make numerous references to Chapters 1 and 2, I have tried to make it easy for the reader to tackle the later chapters without ploughing through the first two. In this way the book can be used as a text for a semester course on metric, normed, and Hilbert spaces. (If

[1] A reference of the form Proposition $(a.b.c)$ is to Proposition c in Section b of Chapter a; one to Exercise $(a.b.c\colon d)$ is to the dth exercise in the set of exercises with reference number $(a.b.c)$; and one to (B3) is to the 3rd result in Appendix B. Within each section, displays that require reference indicators are numbered in sequence: $(1),(2),\ldots$. The counter for this numbering is reset at the start of a new section.

Chapter 2 is not covered, the instructor may need to omit material that depends on familiarity with the Lebesgue integral—in particular Section 4 of Chapter 4.) Chapter 6 could be included to round off an introductory course on functional analysis.

Chapter 1 could be used on its own as a second course on real analysis (following the typical advanced calculus course that introduces formal notions of convergence and continuity); it could also be used as a first course for senior students who have not previously encountered rigorous analysis. Chapters 1 and 2 together would make a good course on real variables, in preparation for either the material in Chapters 3 through 5 or a course on measure theory. The whole book could be used for a sequence of courses starting with real analysis and culminating in an introduction to functional analysis.

I have drawn on the resource provided by many excellent existing texts cited in the bibliography, as well as some original papers (notably [39], in which Riesz introduced the development of the Lebesgue integral used in Chapter 2). My first drafts were prepared using the T^3 *Scientific Word Processing System*; the final version was produced by converting the drafts to TEX and then using *Scientific Word.* Both T^3 and *Scientific Word* are products of TCI Software Research, Inc.

I am grateful to the following people who have helped me in the preparation of this book:

— Patrick Er, who first suggested that I offer a course in analysis for economists, which mutated into the regular analysis course from which the book eventually emerged;

— the students in my analysis classes from 1990 to 1996, who suffered various slowly improving drafts;

— Cris Calude, Nick Dudley Ward, Mark Schroder, Alfred Seeger, Doru Stefanescu, and Wang Yuchuan, who read and commented on parts of the book;

— the wonderfully patient and cooperative staff at Springer–Verlag;

— my wife and children, for their patience (in more than one sense).

It is right and proper for me here to acknowledge my unspoken debt of gratitude to my parents. This book really began 35 years ago, when, with their somewhat mystified support and encouragement, I was beginning my love affair with mathematics and in particular with analysis. It is sad that they did not live to see its completion.

Douglas Bridges
28 January 1997

Contents

Introduction

We may our ends by our beginnings know.
OF PRUDENCE (Sir John Denham)

What we now call *analysis* grew out of the calculus of Newton and Leibniz, was developed throughout the eighteenth century (notably by Euler), and slowly became logically sound (rigorous) through the work of Gauss, Cauchy, Riemann, Weierstrass, Lebesgue, and many others in the nineteenth and early twentieth centuries.

Roughly, analysis may be characterised as the study of limiting processes within mathematics. These processes traditionally include the convergence of infinite sequences and series, continuity, differentiation, and integration, on the real number line $\mathbf{R}$; but in the last 100 years analysis has moved far from the one– or finite–dimensional setting, to the extent that it now deals largely with limiting processes in infinite–dimensional spaces equipped with structures that produce meaningful abstractions of such notions as *limit* and *continuous*. Far from being merely the fantastical delight of mathematicians, these infinite–dimensional abstractions have served both to clarify phenomena whose true nature is often obscured by the peculiar structure of $\mathbf{R}$, and to provide foundations for quantum physics, equilibrium economics, numerical approximation—indeed, a host of areas of pure and applied mathematics. So important is analysis that it is no exaggeration to describe as seriously deficient any honours graduate in physics, mathematics, or theoretical economics who has not had good exposure to at least the fundamentals of metric, normed, and Hilbert space theory, if not the next step, in which metric notions all but disappear in the further abstraction of topological spaces.

Like many students of mathematics, even very good ones, you may find it hard to see the point of analysis, in which intuition often seems sacrificed to the demon of rigour. Is our intuition—algebraic, arithmetic, and geometric—not a sufficiently good guide to mathematical reality in most cases? Alas, it is not, as is illustrated by considering the differentiability of functions. (We are assuming here that you are familiar with the derivative from elementary calculus courses.)

When you first met the derivative, you probably thought that any continuous (real–valued) function—that is, loosely, one with an unbroken graph—on an interval of **R** has a derivative at all points of its domain; in other words, its graph has a tangent everywhere. Once you came across simple examples, like the absolute value function $x \mapsto |x|$, of functions whose graphs are unbroken but have no tangent at some point, it would have been natural to conjecture that if the graph were unbroken, then it had a tangent at all but a finite number of points. If you were really smart, you might even have produced an example of a continuous function, made up of lots of spikes, which was not differentiable at any of a sequence of points. This is about as far as intuition can go. But, as Weierstrass showed in the last century, and as you are invited to demonstrate in Exercise (1.5.1: 2), there exist continuous functions on **R** whose derivative does not exist anywhere. Even this is not the end of the story: in a technical sense discussed in Chapter 6, most continuous functions on **R** are nowhere differentiable! Here, then, is a dramatic failure of our intuition. We could give examples of many others, all of which highlight the need for the sort of careful analysis that is the subject of this book.

Of course, analysis is not primarily concerned with pathological examples such as Weierstrass's one of a continuous, nowhere differentiable function. Its main aim is to build up a body of concepts, theorems, and proofs that describe a large part of the mathematical world (roughly, the continuous part) and are well suited to the mathematical demands of physicists, economists, statisticians, and others. The central chapters of this book, Chapters 3 through 5, give you an introduction to some of the fundamental concepts and results of modern analysis. The earlier chapters serve either as a background reference for the later ones or, if you have not studied much real analysis before, as a rapid introduction to that topic, in preparation for the rest of the book. The final chapter introduces some of the main themes of functional analysis, the study of continuous linear mappings on infinite–dimensional spaces.

Having understood Chapters 3 through 6, you should be in a position to appreciate such other jewels of modern analysis as

- abstract measure spaces, integration, and probability theory;

- approximation theory, in which complicated types of functions are approximated by more tractable ones such as polynomials of fixed maximum degree;
- spectral theory of linear operators on a Hilbert space, generalising the theory of eigenvalues and eigenvectors of matrices;
- analysis of one and several complex variables;
- duality theory in topological vector spaces;
- Haar measure and duality on locally compact groups, and the associated abstract generalisation of the Fourier transform;
- C^*– and von Neumann algebras of operators on a Hilbert space, providing rigorous foundations for quantum mechanics;
- the theory of partial differential equations and the related potential problems of classical physics;
- the calculus of variations and optimisation theory.

These, however, are the subjects of other books. The time has come to begin this one by outlining the background material needed in the main chapters.

Throughout this book, we assume familiarity with the fundamentals of informal set theory, as found in [20]. We use the following notation for sets of numbers.

The set of natural numbers:	$\mathbf{N}$	$=$	$\{0, 1, 2, \ldots\}$.
The set of positive integers:	$\mathbf{N}^+$	$=$	$\{1, 2, 3, \ldots\}$.
The set of integers:	$\mathbf{Z}$	$=$	$\{0, -1, 1, -2, 2, \ldots\}$.
The set of rational numbers:	$\mathbf{Q}$	$=$	$\left\{\pm\frac{m}{n} : m, n \in \mathbf{N},\ n \neq 0\right\}$.

For the purposes of this preliminary section only, we accept as given the algebraic and order properties of the set $\mathbf{R}$ of real numbers, even though these are not introduced formally until Chapter 1.

When the rule and domain describing a function $f : A \to B$ are known or clearly understood, we may denote f by

$$x \mapsto f(x).$$

Note that we use the arrow $\to$ as in "the function $f : A \to B$", and the barred arrow $\mapsto$ as in "the function $x \mapsto x^3$ on $\mathbf{R}$".

We regard two functions with the same rule but different domains as different functions. In fact, we define two functions f and g to be *equal* if and only if