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Preface

An appropriate coverage of the subjects contained in the five parts of this book would need several monographs. We hope that the global treatment presented here may emphasize some of their deep interactions. As far as possible we present self-contained proofs; we have also tried to produce a book that could be used in a graduate course.

Our thread of Ariadne is the introduction into stochastic analysis of the methodology used in classical analysis and differential geometry. Our geometric point of view has obliged us to pay great attention to the foundations. On the other hand our notation, which follows the usual conventions, will allow an experienced worker to look directly at any section of this book, without spending time on the foundational sections.

Each part is constructed according to the following format: a short introduction, a detailed table of contents at the beginning of each chapter of that part and a short note on the literature at the end of each part.

The style of writing oscillates from one part to the next between that of a rather technical monograph in Part II to a broader survey style in Parts IV and V.

I owe a great debt to K. Itô, J.R. Norris, D.W. Stroock for their careful reading of the first draft and for their far-reaching suggestions.

Paris, January 1997

Paul Malliavin

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Part I

Differential Calculus on Gaussian Probability Spaces

In the elementary theory of $\mathbb{R}^n$-valued random variables, operations on the subclass of random variables having a C^1-density relative to Lebesgue measure are often realized through computations of ordinary differential calculus: for instance, the determination of conditional laws by computing differential forms, the realization of a change of variables by computing Jacobians. Our purpose is to extend this methodology to more general probability spaces.

The Lebesgue measure of $\mathbb{R}^n$ can be characterized by its invariance under the group of translations. Given a probability space Ω, *the quasi-automorphism group* will be a "natural" group of transformations of Ω leaving quasi-invariant the probability measure; this notion is quite general; it will be developed in this book in the context of *a Gaussian probability space*, which means an abstract probability space Ω on which we have a Hilbert space $\mathcal{H}$ of Gaussian random variables. The additive group of $\mathcal{H}$ will define the quasi-automorphism group of Ω. Any unitary isomorphism of $\mathcal{H}$ will then generate an automorphism of the Gaussian probability space structure of Ω. The realization of this unitary invariance as a fact built into the construction of Ω itself is done in Chapter I.

The quasi-automorphism group $\mathcal{H}$ operates on a suitable algebra of random variables. The infinitesimal action of $\mathcal{H}$ will lead to the notion of *$\mathcal{H}$-Sobolev spaces* on Ω. Chapter II will be devoted to the study of the algebra of *smooth random variables* which are the random variables belonging to all those Sobolev spaces.

The *Jacobian* of an $\mathbb{R}^d$-valued smooth random variable is defined in Chapter III; an appropriate lower bound for this Jacobian will imply that the corresponding law has a C^∞-density relative to Lebesgue measure. This theorem will result from an interplay between classical harmonic analysis for Sobolev spaces on $\mathbb{R}^d$ and *elliptic estimates* established in Chapter II for Sobolev spaces on Ω. This interplay will be realized by lifting differential forms by the inverse image and pushing down by conditional expectations.

Chapter I

Gaussian Probability Spaces

Contents: Definition of a Gaussian probability space, reducibility – Hermite polynomials on $\mathbb{R}$ – Hermite polynomials on $\mathbb{R}^N$ – Numerical model of a Gaussian probability space – Intrinsic geometry on a Gaussian probability space – The Ornstein-Uhlenbeck semigroup, chaos decomposition – The Cameron-Martin representation – Abstract Wiener space.

In the 1950's Irving Segal developed for the needs of Quantum Field Theory, an *abstract theory of integration of an abstract Hilbert space.* In the 1960's Leonard Gross built the theory of *Gaussian Borel measures* on an arbitrary Banach space. A very special case of Gross's theory is the classical Wiener space that we shall discuss later. Looking in this chapter for the greatest generality combined with the easiest approach, we shall follow a route close to Segal's approach in a version more concrete than his original paper. This point of view is parallel to recent papers of K. Itô.

One basic object of probability theory is the Boolean algebra B of measurable events. By a theorem of Stone this Boolean algebra can be represented as the family of subsets of a compact space K which are both closed and open. The space K can be thought of as the *Stone spectrum* of B; the Stone spectrum is an *intrinsic* object, but one on which we are unable to work: the compact space K is highly non-separable. A substitute for this approach is Gelfand's theory of Banach algebras. We now take for a basic object the algebra of *bounded random variables*; this is a Banach algebra which is isomorphic to the *algebra of continuous functions* on its *Gelfand spectrum.* Again, the Gelfand spectrum is an intrinsically defined compact space; the Gelfand spectrum is highly non-separable.

In order to keep separability within an intrinsic approach, we will introduce not a single model, but a *family* of separable models. The family will have an intrinsic meaning; the choice of a model in the family will lose the intrinsic character. *Intrinsic properties* will then be defined as properties which are independent of the choice of the model.

Some readers might wish to avoid the axiomatic method developed in this chapter. They should then proceed in the following way: skip the reading of the first seven sections and go directly to the last section where the classical abstract

Wiener space appears as an example; this example in mind, it is possible to follow the main lines of the whole book.

1. Axioms of Gaussian probability spaces

1.1 Definition. A *Gaussian probability space* $(\Omega, \mathcal{A}, \mathcal{P}; H, \mathcal{A}_H^{\perp})$ is given by the following elements

1.1.1. $(\Omega, \mathcal{A}, \mathcal{P})$ a probability space.

1.1.2. A *closed subspace* H of $L^2(\Omega, \mathcal{A}, \mathcal{P})$ such that all the random variables belonging to H have a *centered Gaussian law.*

The σ-field generated by those variables is denoted by $\mathcal{A}_H$.

1.1.3. Another σ-field $\mathcal{A}_H^{\perp}$ independent of $\mathcal{A}_H$ and such that

$$\mathcal{A}_H \otimes \mathcal{A}_H^{\perp} = \mathcal{A}.$$

We shall call $\mathcal{A}_H^{\perp}$ the σ-field of the transverse variables or the transverse σ-field.

1.2. Subspaces

Given a Gaussian probability space $(\Omega, \mathcal{A}, \mathcal{P}; H, \mathcal{A}_H^{\perp})$, a *subspace* will be given by

1.2.1. $H' \subset H$ a closed subspace of H.

1.2.2. $\mathcal{A}_{H'}^{\perp}$ a sub σ-field of $\mathcal{A}$ such that $\mathcal{A}_{H'}$ and $\mathcal{A}_{H'}^{\perp}$ are independent. We denote $\mathcal{A}' = \mathcal{A}_{H'} \otimes \mathcal{A}_{H'}^{\perp}$, and we assume

$$\mathcal{A}' \cap \mathcal{A}_H^{\perp} \subset \mathcal{A}_{H'}^{\perp}.$$

1.2.3. Example. Given a Gaussian probability space $(\Omega, \mathcal{A}, \mathcal{P}, H, \mathcal{A}_H^{\perp})$ and given a closed vector subspace H' of H, then there exists $\mathcal{B}$ such that $(\Omega, \mathcal{A}, \mathcal{P}, H', \mathcal{B})$ is a subspace of $(\Omega, \mathcal{A}, \mathcal{P}, H, \mathcal{A}_H^{\perp})$.

Proof. Denote by V the orthogonal complement of H' in H. Then orthogonality in H implies independence. Therefore $\mathcal{A}_V$ is independent of $\mathcal{A}_{H'}$. We take for $\mathcal{B}$ the σ-field generated by $\mathcal{A}_V$ and $\mathcal{A}_H^{\perp}$. □

1.3. Irreducibility

1.3.1. Definition. A Gaussian probability space is *irreducible* if $\mathcal{A}_H = \mathcal{A}$.

1.3.2. Remark. At first sight it might appear strange to work with non-irreducible spaces. This concept is introduced for the following reasons:

(i) The possibility to work with a subspace (as explained in 1.2.3) which will provide approximations by *finite dimensional* Gaussian spaces.
(ii) The general idea that a probability space can always be extended and that the admissible operations have to be stable under the extension of the space.

1.3.3. Remark. We introduce the Hilbert space $G = L^2(\Omega, \mathcal{A}_H^\perp, \mathcal{P})$. Then

$$L^2(\Omega, \mathcal{A}, \mathcal{P}) = L^2\big((\Omega, \mathcal{A}_H, \mathcal{P});\ G\big),$$

where the r.h.s. denotes an L^2 space of G-valued functions.

1.4. Isomorphism

Given two Gaussian spaces $(\Omega, \mathcal{A}, \mathcal{P}, H)$ and $(\Omega', \mathcal{A}', \mathcal{P}', H')$, an isomorphism will be given by an isometry $u : L^2(\Omega', \mathcal{A}', \mathcal{P}') \to L^2(\Omega, \mathcal{A}, \mathcal{P})$ such that u restricted to L^∞ is an algebraic homomorphism and such that $u(H') = H$, $u\left(L^2\left(\Omega', \mathcal{A}_{H'}^\perp, \mathcal{P}'\right)\right) = L^2\left(\Omega, \mathcal{A}_H^\perp, \mathcal{P}\right)$.

Example 1. A bijective measurable map $j : \Omega \to \Omega'$ preserving the probability measure, the spaces H and H', and $\mathcal{A}_H^\perp$ and $\mathcal{A}_{H'}^\perp$, induces such an isomorphism. If the σ-fields $\mathcal{A}$ and $\mathcal{A}'$ are "sufficiently large" in Ω and Ω', all the isomorphisms are of this nature. (This is the case if Ω, Ω' are separable, complete metric spaces and if $\mathcal{A}$ and $\mathcal{A}'$ contain the Borel σ-fields.)

One of the purposes of this chapter is to show that *the equivalence classes under isomorphism of irreducible Gaussian probability spaces* $(\Omega, \mathcal{A}, \mathcal{P}, H)$ *are classified by the dimension of* H. To prepare the proof of this result, we shall recall in the next two subsections some notation and classical results on Hermite polynomials.

2. Hermite polynomials on $\mathbb{R}$

We consider on $\mathbb{R}$ the normal law

$$\gamma(d\xi) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\xi^2}{2}\right) d\xi.$$

We consider the real Hilbert space associated to the scalar product

$$\int \psi(\xi)\varphi(\xi)\gamma(d\xi) = (\psi \mid \varphi).$$

We denote by $C_0^1(\mathbb{R})$ the set of C^1-functions with compact support. We denote by ∂ the operator of differentiation:

$$(\partial\varphi)(\xi) = \varphi'(\xi).$$

Remark. In mathematical physics the operator ∂ is called the *annihilation operator.*

2.1. Lemma. *Denote by ∂^* the operator defined, for $\varphi \in C^1(\mathbb{R})$, by*

$$(\partial^*\varphi)(\xi) = -\varphi'(\xi) + \xi\varphi(\xi).$$

Then if $\partial\varphi$ and $\partial^*\psi \in L^2(\gamma)$ we have

$$(\partial\varphi \mid \psi) = (\varphi \mid \partial^*\psi).$$

Remark. In mathematical physics, the operator ∂^* is called the *creation operator.*

Proof. If φ has compact support, this identity is obtained by integration by parts. Set $q(\xi) = (1 - |\xi|)^+$, $q_\varepsilon(\xi) = q(\varepsilon\xi)$.Then

$$(\partial(q_\varepsilon\varphi) \mid \psi) = (q_\varepsilon\varphi \mid \partial^*\psi)$$

and when $\varepsilon \to 0$ the right hand side converges.

The same property of the left hand side will result from $\lim ((\partial q_\varepsilon)\varphi \mid \psi) = 0$, which follows from the estimate

$$|((\partial q_\varepsilon)\varphi \mid \psi)| < \varepsilon\|\varphi\|_{L^2}\|\psi\|_{L^2}.$$

□

2.2. Definition of the Hermite polynomials. We define the sequence

$$\begin{aligned} H_0(\xi) &= 1 \\ H_n &= \partial^* H_{n-1} = (\partial^*)^n 1. \end{aligned}$$

By induction on n we see that H_n is a polynomial of degree n and that its term of highest degree is ξ^n.

2.3. Lemma.

$$\partial\partial^* - \partial^*\partial = 1.$$

Proof. A straightforward computation establishes this commutation relation, which is basic in quantum mechanics. □

2.4. Lemma. *Denote by μ a probability measure on $\mathbb{R}^n$ such that there exists $c > 0$ for which*

$$\int_{\mathbb{R}^n} e^{c|y|}\mu(dy) < +\infty$$

then the polynomials in the coordinate functions are dense in $L^p(\mu)$ for $p \in [1, +\infty)$.

Proof. Denote by V the L^p-closure of the polynomials. If $V \neq L^p$ we can find $u \in L^q(\mu)$, $u \neq 0$, such that u is orthogonal to V. Consider the formal integral

$$\hat{u}(t) = \int_{\mathbb{R}^n} e^{it.y} u(y)\mu(dy).$$

Now

$$\left|\hat{u}(\sigma + i\tau)\right| \leq \int e^{\tau.y}|u(y)|\mu(dy) \leq \left\|e^{\tau.y}\right\|_{L^p} \|u\|_{L^q} < +\infty$$

which implies the convergence if $|\tau| < \frac{c}{p}$.

Therefore $\hat{u}$ is the restriction to $\mathbb{R}^n$ of a function holomorphic in the tube $\left\{\sigma + i\tau;\ \sigma \in \mathbb{R}^n,\ |\tau| < \frac{c}{p}\right\}$. Furthermore, by the orthogonality to V

$$\left[\frac{\partial}{\partial t_1^{\alpha_1} \ldots \partial t_n^{\alpha_n}} \hat{u}\right](0) = 0.$$

By analyticity this implies $\hat{u} = 0$. By the Fourier inversion formula therefore, $u d\mu = 0$ so $u = 0$ a.e. μ, which is a contradiction. □

2.5. Theorem

2.5.1. $\partial H_n = n\ H_{n-1}$.

2.5.2. $(n!)^{-1/2} H_n$ *is an orthonormal basis of* $L^2(\gamma)$.

2.5.3. *Define* $\mathcal{L} = \partial^*\partial$, *then* $\mathcal{L}H_n = n\ H_n$.

2.5.4. *Let* $f \in L^2(\gamma)$. *Assume that all derivatives of* f *belong to* $L^2(\gamma)$, *then the* L^2*-expansion of* f *can be written*

$$f = \sum_{n=0}^{+\infty} \frac{1}{n!} E\left(\partial^n f\right) H_n$$

where $E(u)$ *denotes* $(u \mid 1)$.

2.5.5.

$$\int_{\mathbb{R}^2} H_n(\xi \cos\theta + \eta \sin\theta) H_p(\xi)\gamma(d\xi)\gamma(d\eta) = n!(\cos\theta)^n \delta_n^p$$

(where $\delta_n^p = 0$ *if* $p \neq n$ *and* $\delta_n^n = 1$).

2.5.6.

$$\exp\left(\lambda x - \frac{\lambda^2}{2}\right) = \sum_{n=0}^{+\infty} \frac{\lambda^n}{n!} H_n(x),$$

this convergence being uniform on compact sets in (λ, x). *This identity is called the generating function identity.*

Proof.

(2.5.1) For $n = 1$, $H_1 = \xi$, $\partial H_1 = 1 = H_0$; we proceed by induction on n. Assume that (i) is true for $n < p$. Then

$$\begin{aligned} \partial H_p &= \partial\partial^* H_{p-1} = \partial^*\partial H_{p-1} + H_{p-1}, \quad \text{now by 1.2.2} \\ &= \partial^*(p-1)H_{p-2} + H_{p-1} = pH_{p-1}. \end{aligned}$$

□

(2.5.2)

$$\left(H_s \mid (\partial^*)^\ell 1\right) = (\partial^\ell H_s \mid 1)$$

If $\ell > s$ as H_s is a polynomial of degree s, we have $\partial^\ell H_s = 0$. If $\ell = s$, then $\partial^s H_s = s!$. This proves that $\left\{\frac{1}{(s!)^{1/2}} H_s\right\}$ is an orthonormal system. Therefore $\{H_s\}$ are linearly independent. They generate by linear combination the vector space of polynomials which is dense in L^2 by 2.4. □

(2.5.3)

$$\partial^*\partial H_n = \partial^* n H_{n-1} = n\partial^* H_{n-1} = nH_n$$

□

(2.5.4) By (ii) every $f \in L^2(\gamma)$ has the following L^2 expansion:

$$f = \sum c_n H_n \quad \text{with} \quad c_n = \frac{1}{n!}(f \mid H_n).$$

As f and f' belong to $L^2(\gamma)$ we have by 2.1

$$(f \mid H_n) = \left(f \mid \partial^* H_{n-1}\right) = (\partial f \mid H_{n-1})$$

and so by induction

$$\dots = \left(\partial^{n-1} f \mid H_1\right) = \left(\partial^{n-1} f \mid \partial^* 1\right) = \left(\partial^n f \mid 1\right).$$

□

(2.5.5) We remark that $\gamma \otimes \gamma$ is the normal law $\gamma_{\mathbb{R}^2}$ on $\mathbb{R}^2$, therefore it is rotation invariant. We denote by ∂_θ the derivation operator defined on smooth functions on $\mathbb{R}^2$ by

$$(\partial_\theta f) = \cos\theta \frac{\partial f}{\partial \xi} + \sin\theta \frac{\partial f}{\partial \eta}.$$

We denote by ∂_θ^* the adjoint of ∂_θ in $L^2(\gamma_{\mathbb{R}^2})$.