

$$\delta({}_t\bar{V}) + \pi(t) - \mu$$

$$\frac{d}{dt}$$

Fundamentals of Actuarial Mathematics

Third Edition

S. David Promislow

$$\frac{d}{dt}{}_t\bar{V} = \delta({}_t\bar{V}) + \pi(t) - \mu_x(t)(b(t) - {}_t\bar{V})$$

$$\delta({}_t\bar{V}) +$$

$$+ \pi(t) - \mu_x(t)(b(t) -$$

$$\mu_x$$

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Fundamentals of Actuarial Mathematics

Third Edition

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Fundamentals of Actuarial Mathematics

To Georgia and Griffith

Preface

The third edition of this book continues the objective of providing coverage of actuarial mathematics in a flexible manner that meets the needs of several audiences. These range from those who want only a basic knowledge of the subject, to those preparing for careers as professional actuaries. All this is carried out with a streamlined system of notation, and a modern approach to computation involving spreadsheets.

The text is divided into four parts. The first two cover the subject of life contingencies. The modern approach towards this subject is through a stochastic model, as opposed to the older deterministic viewpoint. I certainly agree that mastering the stochastic model is the desirable goal. However, my classroom experience has convinced me that this is not the right place to begin the instruction. I find that students are much better able to learn the new ideas, the new notation, the new ways of thinking involved in this subject, when done first in the simplest possible setting, namely a deterministic discrete model. After the main ideas are presented in this fashion, continuous models are introduced. In Part II of the book, the full stochastic model of life contingencies can be dealt with in a reasonably quick fashion.

Another innovation in Part II is to depart from the conventional treatment of life contingencies as dealing essentially with patterns of mortality or disability in a group of human lives. Throughout Part II, we deal with general *failure times* which makes the theory more widely adaptable.

Part III deals with more advanced stochastic models. Following an introduction to stochastic processes, there is a chapter covering multi-state theory, an approach which unifies many of the ideas in Parts I and II. The final chapter in Part III is an introduction to modern financial mathematics.

Part IV deals with the subject of risk theory, sometime referred to as loss models. It includes an extensive coverage of classical ruin theory, a topic that originated in actuarial science but recently has found many applications in financial economics. It also includes credibility theory, which will appeal to the reader interested more in the casualty side of actuarial mathematics.

This book will meet the needs of those preparing for the examinations of many of the major professional actuarial organizations. Parts I to III of this new third edition covers all of the material on the current syllabuses of Exam MLC of the Society of Actuaries and Canadian Institute of Actuaries and Exam LC of the Casualty Actuarial Society, and covers most of the topics on the current syllabus of Exam CT5 of the British Institute of Actuaries.

In addition, Part IV of the book covers a great deal of the material on Exam C of the Society of Actuaries and Canadian Institute of Actuaries, including the topics of Frequency, Severity and Aggregate Models, Risk Measures, and Credibility Theory.

The mathematical prerequisites for Part I are relatively modest, comprising elementary linear algebra and probability theory, and, beginning in Chapter 8, some basic calculus. A more advanced knowledge of probability theory is needed from Chapter 13 onward, and this material summarized in Appendix A. A usual prerequisite for actuarial mathematics is a course in the theory of interest. Although this may be useful, it is not strictly required. All the interest theory that is needed is presented as a particular case of the general deterministic actuarial model in Chapter 2.

A major source of difficulty for many students in learning actuarial mathematics is to master the rather complex system of actuarial notation. We have introduced some notational innovations, which tie in well with modern calculation procedures as well as allow us to greatly simplify the notation that is required. We have, however, included all the standard notation in separate sections, at the end of the relevant chapters, which can be read by those readers who desire this material.

Keeping in mind the nature of the book and its intended audience, we have avoided excessive mathematical rigour. Nonetheless, careful proofs are given in all cases where these are thought to be accessible to the typical senior undergraduate mathematics student. For the few proofs not given in their entirety, mainly those involving continuous-time stochastic processes, we have tried at least to provide some motivation and intuitive reasoning for the results.

Exercises appear at the end of each chapter. In Parts I and II these are divided up into different types. Type A exercises generally are those which involve direct calculation from the formulas in the book. Type B involve problems where more thought is involved. Derivations and problems which involve symbols rather than numeric calculation are normally included in Type B problems. A third type is spreadsheet exercises which themselves are divided into two subtypes. The first of these asks the reader to solve problems using a spreadsheet. Detailed descriptions of applicable Microsoft Excel[®] spreadsheets are given at the end of the relevant chapters. Readers of course are free to modify these or construct their own. The second subtype does not ask specific questions but instead asks the reader to modify the given spreadsheets to handle additional tasks. Answers to most of the calculation-type exercises appear at the end of the book.

Sections marked with an asterisk * deal with more advanced material, or with special topics that are not used elsewhere in the book. They can be omitted on first reading. The exercises dealing with such sections are likewise marked with *, as are a few other exercises which are of above average difficulty.

There are various ways of using the text for university courses geared to third or fourth year undergraduates, or beginning graduate students. Chapters 1 to 8 could form the basis of a one-semester introductory course. Part IV is for the most part independent of the first three parts, except for the background material on stochastic processes given in Chapter 18 and would constitute another one-semester course. The rest of the book constitutes roughly another two semesters worth of material, with possibly some omissions; Chapter 13 is not needed for the rest of the book. Chapters 7 (except for Section 7.3.1), 9 and 12 deal with topics that are important in applications, but which are used minimally in other parts of the text. They could be omitted without loss of continuity.

CHANGES IN THE THIRD EDITION

There are several additions and changes to the third edition.

The most notable is a new Chapter 20 providing an introduction to the mathematics of financial markets. It has been long recognized that knowledge of this subject is essential to the management of financial risk that faces the actuary of today.

Other additions include the following:

- Chapter 12, on expenses, has been considerably enlarged to include the topic of profit testing.
- The chapter on multi-state models has been expanded to include discussion of reserves and profit testing in such models, as well as several additional techniques for continuous-time problems.
- Some extra numerical procedures have been included, such as Euler's method for differential equations, and the three-term Woolhouse formulas for fractional annuity approximations.
- An introduction to Brownian motion has been added to the material on continuous-time stochastic processes.
- The previous material on universal life and variable annuities has been rewritten and included in a new chapter dealing with miscellaneous topics. A brief discussion of pension plans is included here as well.
- Additional examples, exercises, and clarification have been added to various chapters.

As well as the changes there has been a reorganization in the material. The previous two chapters on stochastic processes have been combined into one and now appear earlier in the book as background for the multi-state and financial markets chapters. In the current Part IV, the detailed descriptions of the various distributions have been removed and added as a section to the Appendix on probability theory.

Acknowledgements

Several individuals have assisted in the various editions of this book. I am particularly indebted to two people who have made a significant contribution by providing a number of helpful comments, corrections, and suggestions. They are Virginia Young for her work on the first edition, and Elias Shiu for his help with the third edition.

There are many others who deserve thanks. Moshe Milevsky provided enlightening comments on annuities and it was his ideas that motivated the credit risk applications in Chapter 10, as well as some of the material on generational annuity tables in Chapter 9. Several people found misprints in the first edition and earlier drafts. These include Valerie Michkine, Jacques Labelle, Karen Antonio, Kristen Moore, as well as students at York University and the University of Michigan. Christian Hess asked some questions which led to the inclusion of Example 21.10 to clear up an ambiguous point. Exercise 18.13 was motivated by Bob Jewett's progressive practice routines for pool. My son Michael, a life insurance actuary, provided valuable advice on several practical aspects of the material. Thanks go to the editorial and production teams at Wiley for their much appreciated assistance. Finally, I thank my wife Shirley who provided support and encouragement throughout the writing of all three editions.

About the companion website

This book is accompanied by a companion website:

www.wiley.com/go/promislow/actuarial

The website includes:

- A variety of exercises, both computational and theoretical
- Answers, enabling use for self-study.

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