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Special Functions and Orthogonal Polynomials

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SPECIAL FUNCTIONS AND ORTHOGONAL POLYNOMIALS

The subject of special functions is often presented as a collection of disparate results, rarely organized in a coherent way. This book emphasizes general principles that unify and demarcate the subjects of study. The authors' main goals are to provide clear motivation, efficient proofs, and original references for all of the principal results.

The book covers standard material, but also much more. It shows how much of the subject can be traced back to two equations – the hypergeometric equation and the confluent hypergeometric equation – and it details the ways in which these equations are canonical and special. There is extended coverage of orthogonal polynomials, including connections to approximation theory, continued fractions, and the moment problem, as well as an introduction to new asymptotic methods. The book includes chapters on Meijer G-functions and elliptic functions. The final chapter introduces Painlevé transcendents, which have been termed the "special functions of the twenty-first century."

Richard Beals was Professor of Mathematics at the University of Chicago and at Yale University. He is the author or co-author of books on mathematical analysis, linear operators, and inverse scattering theory, and has authored more than a hundred research papers in areas including partial differential equations, mathematical economics, and mathematical psychology.

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Preface

This book originated as *Special Functions: A Graduate Text*. The current version is considerably enlarged: the number of chapters devoted to orthogonal polynomials has increased from two to four; Meijer G-functions and Painlevé transcendents are now treated.

As we noted in the earlier book, the subject of special functions lacks a precise delineation, but it has a long and distinguished history. The remarks at the end of each chapter discuss the history, with numerous references and suggestions for further reading.

This book covers most of the standard topics and some that are less standard. We have tried to provide context for the discussion by emphasizing unifying ideas. The text and the problems provide proofs or proof outlines for nearly all the results and formulas.

We have also tried to keep the prerequisites to a minimum: a reasonable familiarity with power series and integrals, convergence, and the like. Some proofs rely on the basics of complex function theory, which are reviewed in the first appendix. Some familiarity with Hilbert space ideas, in the L^2 framework, is useful. The chapters on elliptic functions and on Painlevé transcendents rely more heavily than the rest of the book on concepts from complex analysis. The second appendix contains a quick development of basic results from Fourier analysis, including the Mellin transform.

The first chapter provides a general context for the discussion of the linear theory, especially in connection with special properties of the hypergeometric and confluent hypergeometric equations. Chapter 2 treats the gamma and beta functions at some length, with an introduction to the Riemann zeta function. Chapter 3 covers the relevant material from the theory of ordinary differential equations, including a characterization of the classical polynomials as eigenfunctions, and a discussion of separation of variables for equations involving the Laplacian.

The next four chapters are concerned with orthogonal polynomials on a real interval. Chapter 4 introduces the general theory, including three-term recurrence relations, Padé approximants, continued fractions, and Favard's theorem. The classical polynomials (Hermite, Laguerre, Jacobi) are treated in detail in Chapter 5, including asymptotic distribution of zeros. Chapter 6 introduces finite difference analogues of the classification theorem, yielding the classical discrete polynomials as well as neoclassical versions and the Askey scheme. Two methods of obtaining asymptotic results are presented in Chapter 7. In particular, the Riemann–Hilbert method is carried through for Hermite polynomials.

Chapters 8 through 11 contain a detailed treatment of the confluent hypergeometric equation, the hypergeometric equation, and special cases. These include Weber functions, Whittaker functions, Airy functions, cylinder functions (Bessel, Hankel, ...), spherical harmonics, and Legendre functions. Among the topics are linear relations, various transformations, integral representations, and asymptotics. Chapter 13 contains proofs of asymptotic results for these functions and for the classical polynomials.

In Chapter 12 we extend an earlier discussion of the special "recursive" property of the hypergeometric and confluent hypergeometric equations to equations of arbitrary order. This property characterizes the generalized hypergeometric equation. The corresponding solutions, the generalized hypergeometric functions, are covered in more detail than in the first version. Elliptic integrals, elliptic functions of Jacobi and Weierstrass, and theta functions are treated in Chapter 14.

The principal new topics, Meijer *G*-functions and Painlevé transcendents, have current theoretical and practical interest.

Meijer *G*-functions, which are special solutions of generalized hypergeometric equations, are introduced in Chapter 12. They generalize the classic Mellin–Barnes integral representations. The *G*-functions occur in probability and physics, and play a large role in compiling tables of integrals.

Chapter 15 has an extensive introduction to the classical and modern theory of Painlevé equations and their solutions, with emphasis on the second Painlevé equation, PII. Painlevé's method is introduced and PII is derived in detail. The isomonodromy method and Bäcklund transformations are introduced, and used to obtain rational solutions and information about general solutions. The Riemann–Hilbert method is used to derive a connection formula for solutions of PII(0). Applications include differential geometry, random matrix theory, integrable systems, and statistical physics.

The earlier book contained a concise summary of each chapter. These have been omitted here, partly to save space, and partly because the summaries often proved to be more annoying than helpful in use of the book for reference.

The first-named author acknowledges the efforts of some of his research collaborators, especially Peter Greiner, Bernard Gaveau, Yakar Kannai, David

Sattinger, and Jacek Szmigielski, who managed over a period of years to convince him that special functions are not only useful but beautiful.

The authors thank Jacek Szmigielski, Mourad Ismail, Richard Askey, and an anonymous reviewer for helpful comments on the earlier manuscript. The first-named author is grateful to the Department of Mathematics and to the Liu Bie Ju Centre for Mathematical Sciences at City University of Hong Kong for help and hospitality during the preparation of both versions of this book. The second-named author is happy to acknowledge all his former students, collaborators, and assistants who helped with this project.

Contents

1

Orientation

The concept of a "special function" has no precise definition. From a practical point of view, a special function is a function of one variable that is (a) not one of the "elementary functions" – algebraic functions, trigonometric functions, the exponential, the logarithm, and functions constructed algebraically from these functions – and is (b) a function about which one can find information in many of the books about special functions. A large amount of such information has been accumulated over a period of three centuries. Like such elementary functions as the exponential and trigonometric functions, special functions come up in numerous contexts. These contexts include both pure mathematics and applications, ranging from number theory and combinatorics to probability and physical science.

The majority of special functions that are treated in many of the general books on the subject are solutions of certain second-order linear differential equations. Indeed, these functions were discovered through the study of physical problems: vibrations, heat flow, equilibrium, and so on. The associated equations are partial differential equations of second-order. In some coordinate systems these equations can be solved by separation of variables, leading to the second-order ordinary differential equations in question. (Solutions of the analogous *first-order* linear differential equations are elementary functions.)

Despite the long list of adjectives and proper names attached to this class of special functions (hypergeometric, confluent hypergeometric, cylinder, parabolic cylinder, spherical, Airy, Bessel, Hankel, Hermite, Kelvin, Kummer, Laguerre, Legendre, Macdonald, Neumann, Weber, Whittaker, ...), each of them is closely related to one of two families of equations: the confluent hypergeometric equation(s)

$$x\,u''(x) + (c - x)\,u'(x) - a\,u(x) = 0 \tag{1.0.1}$$

and the hypergeometric equation(s)

$$x(1-x)\,u''(x) + [c - (a+b+1)x]\,u'(x) - ab\,u(x) = 0. \tag{1.0.2}$$

The parameters a, b, c are real or complex constants.

Some solutions of these equations are polynomials: up to a linear change of variables, they are the "classical orthogonal polynomials." Again there are many names attached: Chebyshev, Gegenbauer, Hermite, Jacobi, Laguerre, Legendre, ultraspherical. In this introductory chapter we discuss one context in which these equations, and (up to normalization) no others, arise. We also shall see how two equations can, in principle, give rise to such a menagerie of functions.

Some special functions are *not* closely connected to linear differential equations. These exceptions include the gamma function, the beta function, elliptic functions, and the Painlevé transcendents.

The gamma and beta functions evaluate certain integrals. They are indispensable in many calculations, especially in connection with the class of functions mentioned earlier, as we illustrate below.

Elliptic functions arise as solutions of a simple *nonlinear* second-order differential equation, and also in connection with integrating certain algebraic functions. They have a wide range of applications, from number theory to integrable systems.

The Painlevé transcendents are solutions of a class of nonlinear second-order equations that share a crucial property with the equations that characterize elliptic functions, in that the solutions are single-valued in certain fixed domains, independent of the initial conditions.

1.1 Power series solutions

The general homogeneous linear second-order equation is

$$p(x)\,u''(x) + q(x)\,u'(x) + r(x)\,u(x) \;=\; 0, \tag{1.1.1}$$

with p not identically zero. We assume here that the coefficient functions p, q, and r are holomorphic (analytic) in a neighborhood of the origin.

If a function u is holomorphic in a neighborhood of the origin, then the function on the left side of (1.1.1) is also holomorphic in a neighborhood of the origin. The coefficients of the power series expansion of this function can be computed from the coefficients of the expansions of the functions p, q, r, and u. Under these assumptions, (1.1.1) is equivalent to the sequence of equations obtained by setting the coefficients of the expansion of the left side equal to zero. Specifically, suppose that the coefficient functions p, q, r have series expansions

$$p(x) = \sum_{k=0}^{\infty} p_k x^k, \quad q(x) = \sum_{k=0}^{\infty} q_k x^k, \quad r(x) = \sum_{k=0}^{\infty} r_k x^k,$$

and u has the expansion

$$u(x) = \sum_{k=0}^{\infty} u_k x^k.$$

Then the constant term and the coefficients of x and x^2 on the left side of (1.1.1) are

$$\begin{aligned}
&2p_0u_2 + q_0u_1 + r_0u_0, &(1.1.2)\\
&6p_0u_3 + 2p_1u_2 + 2q_0u_2 + q_1u_1 + r_1u_0 + r_0u_1,\\
&12p_0u_4 + 6p_1u_3 + 2p_2u_2 + 3q_0u_3 + 2q_1u_2 + q_2u_1 + r_0u_2 + r_1u_1 + r_2u_0,
\end{aligned}$$

respectively. The sequence of equations equivalent to (1.1.1) is the sequence

$$\sum_{j+k=n,\,k\geq 0} (k+2)(k+1)p_j u_{k+2} + \sum_{j+k=n,\,k\geq 0} (k+1)q_j u_{k+1} + \sum_{j+k=n,k\geq 0} r_j u_k = 0, \qquad n = 0, 1, 2, \ldots \tag{1.1.3}$$

We say that (1.1.1) is *recursive* if it has a nonzero solution u that is holomorphic in a neighborhood of the origin, and the equations (1.1.3) determine the coefficients $\{u_n\}$ by a simple recursion: the nth equation determines u_n in terms of u_{n-1} alone. Suppose that (1.1.1) is recursive. Then the first of the equations (1.1.2) should involve u_1 but not u_2, so $p_0 = 0$, $q_0 \neq 0$. The second equation should not involve u_3 or u_0, so $r_1 = 0$. Similarly, the third equation shows that $q_2 = r_2 = 0$. Continuing, we obtain

$$p_0 = 0, \quad p_j = 0,\ j \geq 3; \quad q_j = 0,\ j \geq 2; \quad r_j = 0,\ j \geq 1.$$

Collecting terms, we see that the nth equation is

$$[(n+1)np_1 + (n+1)q_0]\, u_{n+1} + [n(n-1)p_2 + nq_1 + r_0]\, u_n = 0.$$

For special values of the parameters p_1, p_2, q_0, q_1, r_0, one of these coefficients may vanish for some value of n. In such a case, either the recursion breaks down, or the solution u is a polynomial. We assume that this does not happen. Thus

$$u_{n+1} = -\frac{n(n-1)p_2 + nq_1 + r_0}{(n+1)np_1 + (n+1)q_0}\, u_n. \tag{1.1.4}$$

Assume $u_0 \neq 0$. If $p_1 = 0$ but $p_2 \neq 0$, the series $\sum_{n=0}^{\infty} u_n x^n$ diverges for all $x \neq 0$ (ratio test). Therefore, up to normalization – a linear change of coordinates and a multiplicative constant – we may assume that $p(x)$ has one of the two forms $p(x) = x(1-x)$ or $p(x) = x$.

If $p(x) = x(1-x)$, then (1.1.1) has the form

$$x(1-x)\,u''(x) + (q_0 + q_1 x)\,u'(x) + r_0\,u(x) = 0.$$

Constants a and b can be chosen so that this becomes (1.0.2).

If $p(x) = x$ and $q_1 \neq 0$ we may replace x by a multiple of x and take $q_1 = -1$. Then (1.1.1) has the form (1.0.1).

Finally, suppose $p(x) = x$ and $q_1 = 0$. If also $r_0 = 0$, then (1.1.1) is a first-order equation for u'. Otherwise, we may replace x by a multiple of x and take $r_0 = 1$. Then (1.1.1) has the form

$$x u''(x) + c u'(x) + u(x) = 0. \tag{1.1.5}$$

This equation is not obviously related to either (1.0.1) or (1.0.2). However, it can be shown that it becomes a special case of (1.0.1) after a change of variable and a "gauge transformation" (see Exercise 5).

In summary: up to certain normalizations, an equation of the form (1.1.1) is recursive if and only if it has one of the three forms (1.0.1), (1.0.2), or (1.1.5). Moreover, (1.1.5) can be transformed to a case of (1.0.1).

Let us note briefly the answer to the analogous question for a homogeneous linear *first-order* equation

$$q(x) u'(x) + r(x) u(x) = 0, \tag{1.1.6}$$

with q not identically zero. This amounts to taking $p = 0$ in the argument above. The conclusion is again that q is a polynomial of degree at most one, with $q_0 \neq 0$, while $r = r_0$ is constant. Up to normalization, $q(x)$ has one of the two forms $q(x) = 1$ or $q(x) = x - 1$. Thus the equation has one of the two forms

$$u'(x) - a u(x) = 0; \qquad (x-1)u'(x) - a u(x) = 0,$$

with solutions

$$u(x) = c e^{ax}, \quad u(x) = c(x-1)^a,$$

respectively.

The analogous question for homogeneous linear equations of *arbitrary* order is taken up in Chapter 12, Section 12.2.

Let us return to the confluent hypergeometric equation (1.0.1). The power series solution with $u_0 = 1$ is sometimes denoted $M(a,c;x)$. It can be calculated easily from the recursion (1.1.4). The result is

$$M(a,c;x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n \, n!} x^n, \quad c \neq 0, -1, -2, \ldots \tag{1.1.7}$$

Here the "shifted factorial" or "Pochhammer symbol" $(a)_n$ is defined by

$$(a)_0 = 1, \qquad (a)_n = a(a+1)(a+2)\cdots(a+n-1), \tag{1.1.8}$$

so that $(1)_n = n!$. The series (1.1.7) converges for all complex x (ratio test), so M is an entire function of x.