ERWIN KREYSZIG

DIFFERENTIAL GEOMETRY

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BY

ERWIN KREYSZIG

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PREFACE

This book provides an introduction to the differential geometry of curves and surfaces in three-dimensional Euclidean space. We first consider some basic concepts and facts of analytic geometry which will be useful for later investigations. The theory of space curves is presented in the second chapter. We then proceed to the foundations of the theory of surfaces. Problems closely related to the first and second fundamental forms are considered in the third and fourth chapter. Chapter V is devoted to geodesics. Several types of mappings of surfaces which are of theoretical or practical importance figure in Chapter VI, including some mappings of the sphere into the plane which are frequently used when constructing maps of the globe. In this connexion different types of special surfaces occur necessarily. This chapter is therefore related to Chapter VIII on special surfaces. The absolute differential calculus and the displacement of Levi-Cività, which is of interest especially in connexion with the theory of relativity, are investigated in Chapter VII. As is natural the results obtained in Chapters III and IV yield the foundations of the Chapters V–VIII.

In the theory of surfaces we make full use of the tensor calculus, which is developed as needed, cf. Sections 27–33. The student will quickly find that this calculus becomes a simple tool as soon as he is accustomed to the few basic concepts and rules, especially to the 'summation convention', cf. Section 27. He will perceive that the tensor method is helpful in achieving a simplification of the analytic formalism of many investigations. Hence tensors are important tools in modern differential geometry.

The presentation in this book may also be considered as a preparation for the Riemannian geometry of n dimensions.

As is well known, tensors are of increasing importance not only in mathematics, but also in the application of mathematics to physics and engineering. Since the problems treated in differential geometry by means of tensor calculus are relatively perspicuous, they enable us to understand not only the formalism but also the nature and essential background of this calculus. The student will thus gain by being able to apply his knowledge of tensors to fields other than that of differential geometry. In using tensor calculus one should never forget that the purpose of this calculus lies in its applications to certain problems; it is a tool only, albeit a very powerful one.

We should mention that many of the topics considered in this book can also be investigated by means of outer differential forms. Cf. E. Cartan, *Les Systèmes différentiels extérieurs et leurs applications géométriques* (Paris, 1945), W. Blaschke, *Einführung in die Differentialgeometrie* (Berlin, 1950).

In writing this book, the experiences which I gained during the period of personal co-operation with Professor H. Behnke (University of Münster in Westphalia), as well as his printed lecture notes on differential geometry, were of help to me. I have tried to present the whole subject-matter in the simplest possible form consistent with the needs of mathematical rigour, and to convey a clear idea of the geometric significance of the different concepts, methods, and results. For this reason also, numerous figures and examples are included in the text.

In order to lessen the reader's difficulties, especially for those who are approaching differential geometry for the first time, the discussion is relatively detailed. The selection of topics included in this book has been made with great care, consideration being given to the didactic point of view as well as the theoretical and practical importance of the different aspects of the subject.

Problems are to be found at the end of almost every section, and the solutions are listed at the end of the book. These exercises should help the reader to become familiar with the material presented in the text and, what is more important, to get acquainted with the manner of reasoning in differential geometry.

Differential geometry has various relations to other fields of mathematics. Besides the calculus other branches, such as function theory, the calculus of variations, and the theory of differential equations, are also basically important in differential geometry. On the other hand, differential geometry is an essential part of the foundations of some applied sciences, for instance physics, geodesy, and geography. Differential geometry has therefore what we may call a 'general character'; I have tried to stress this point of view in connexion with several topics.

This book is a free translation of my *Differentialgeometrie* which appeared in the series *Mathematik und ihre Anwendungen in Physik und Technik* ('Mathematics and its applications to physics and technical science') (Series A, vol. 25) of the Akademische Verlagsgesellschaft, Geest und Portig, Leipzig, Germany. Some minor changes have been made in the course of translation.

Professor H. Behnke (University of Münster) and Professor H. Graf (Technical University of Darmstadt) have made valuable suggestions to me. Professor S. Bergman (Stanford University) and Professor E. Ullrich (University of Giessen) have checked Sections 64 and 84, respectively. Professor C. Loewner (Stanford University), Professor M. Riesz (University of Lund), Professor H. S. M. Coxeter (University of Toronto), Professor P. Scherk (University of Saskatchewan), Professor M. Barner (Technical University of Karlsruhe), and Professor O. Biberstein (University of Ottawa) have read the manuscript carefully, and I have obtained valuable suggestions from all of them in the course of numerous personal discussions. The translated manuscript has been checked by Professor H. S. M. Coxeter, Professor G. F. D. Duff (University of Toronto), Professor J. T. Duprat (University of Ottawa), Professor R. C. Fisher (Ohio State University, Columbus), and Professor L. Sauvè (St. Patrick's College, Ottawa). I wish to express my gratitude to all of them and also to the University of Toronto Press, for their efficient co-operation.

E. K.

CONTENTS

I

PRELIMINARIES

1. Notation. In this section the meaning of some symbols occurring very frequently in our text will be explained and the page on which these symbols are introduced for the first time indicated. The reader will find another index of definitions, formulae, and theorems at the end of this book.

Page 4 x_1, x_2, x_3: Cartesian coordinates in three-dimensional Euclidean space R_3.

9 Bold-face letters **a**, **y**, etc.: Vectors in space R_3; the components of these vectors will be denoted by a_1, a_2, a_3; y_1, y_2, y_3, etc.

28 s arc length of a curve. Derivatives with respect to s will be denoted by dots, e.g.

$$\dot{\mathbf{x}} \equiv \frac{d\mathbf{x}}{ds} \equiv (\dot{x}_1, \dot{x}_2, \dot{x}_3) \equiv \left(\frac{dx_1}{ds}, \frac{dx_2}{ds}, \frac{dx_3}{ds}\right).$$

An arbitrary parameter figuring in the representation of a curve will usually be denoted by t. Derivatives with respect to t will be characterized by primes, e.g.

$$\mathbf{x}' \equiv \frac{d\mathbf{x}}{dt}, \qquad \mathbf{x}'' \equiv \frac{d^2\mathbf{x}}{dt^2}, \quad \text{etc.}$$

29 $\mathbf{t} = \dot{\mathbf{x}}$ unit tangent vector of a curve C: $\mathbf{x}(s)$.

34 $\mathbf{p} = \dfrac{\ddot{\mathbf{x}}}{|\ddot{\mathbf{x}}|}$ unit principal normal vector of that curve.

36 $\mathbf{b} = \mathbf{t} \times \mathbf{p}$ unit binormal vector of that curve.

34 $\kappa = \dfrac{1}{\rho}$ curvature, ρ radius of curvature of a curve.

38 τ torsion of a curve.

75 u^1, u^2 coordinates on a surface.

$$\mathbf{x}_1 \equiv \frac{\partial \mathbf{x}}{\partial u^1}, \qquad \mathbf{x}_2 \equiv \frac{\partial \mathbf{x}}{\partial u^2},$$

$$\mathbf{x}_\alpha \equiv \frac{\partial \mathbf{x}}{\partial u^\alpha}, \qquad \mathbf{x}_{\alpha\beta} \equiv \frac{\partial^2 \mathbf{x}}{\partial u^\alpha \partial u^\beta}, \quad \text{etc.}$$

Page 84 **Summation convention.** If in a product an index figures twice, once as a subscript and once as a superscript, summation must be carried out with respect to that index from 1 to 2; the summation sign will be omitted.

Example: $a^\alpha b_\alpha \equiv \sum_{\alpha=1}^{2} a^\alpha b_\alpha = a^1 b_1 + a^2 b_2$. For further details and examples see Section 27.

82 $d\mathbf{x}\cdot d\mathbf{x} = g_{\alpha\beta}\, du^\alpha du^\beta$: first fundamental form.

86 $g = g_{11} g_{22} - g_{12}^2$: discriminant of the first fundamental form.

90 Superscript: contravariant index.

93 Subscript: covariant index.

104 $g^{\alpha\beta}$ contravariant components of the metric tensor.

107 $\mathbf{n}$ unit normal vector to a surface, $\mathbf{n}_\alpha \equiv \dfrac{\partial \mathbf{n}}{\partial u^\alpha}$, etc.

119 $-d\mathbf{x}\cdot d\mathbf{n} = b_{\alpha\beta}\, du^\alpha du^\beta$: second fundamental form.

122 $\kappa_n = \dfrac{1}{R}$ normal curvature of a surface.

125 $b = b_{11} b_{22} - b_{12}^2$: discriminant of the second fundamental form.

129 κ_1, κ_2 principal curvatures of a surface.

131 K Gaussian curvature of a surface.

131 H mean curvature of a surface.

140 $\Gamma_{\alpha\beta\gamma}$ Christoffel symbols of the first kind.

141 $\Gamma_{\alpha\beta}{}^{\gamma}$ Christoffel symbols of the second kind.

144 $R_{\alpha\beta\gamma\epsilon}$, $R^{\alpha}{}_{\beta\gamma\epsilon}$: components of the curvature tensors.

154 κ_g geodesic curvature.

186 $d\mathbf{n}\cdot d\mathbf{n} = c_{\alpha\beta}\, du^\alpha du^\beta$: third fundamental form.

2. Nature and purpose of differential geometry. In differential geometry properties of geometric configurations (curves, surfaces) are investigated by means of differential and integral calculus. All our considerations will take place in three-dimensional Euclidean space and will, in general, be restricted to real geometric configurations. We will, however, occasionally extend our methods to the complex domain.

A geometric property is called *local*, if it does not pertain to the geometric configuration as a whole but depends only on the form of the configuration in an (arbitrary small) neighbourhood of a point under consideration. For instance, the curvature of a curve is a local property. Since differential geometry is concerned mainly with local properties, it is primarily a *geometry in the small* or a *local geometry.*

This fact does not exclude the possibility of considering geometric configurations as a whole. This kind of investigation belongs to what we call *global differential geometry* or *differential geometry in the large.* In this book we will consider only a small number of global problems, for example, in connexion with the theorem of Gauss–Bonnet. We may say that global problems are problems in which 'macroscopic' properties are related to 'microscopic' ones. For further study in this field see, for example, W. Blaschke, *Vorlesungen über Differentialgeometrie* (3 vols., Berlin, 1945, 1923, and 1929).

As is natural, concepts, methods, and results of analytic geometry will be constantly used in differential geometry. The following sections are consequently devoted to a brief review of some of the topics from analytic geometry which we will need for our further investigations. We may restrict ourselves to the analytic geometry of three-dimensional Euclidean space in which all our considerations will take place.

3. Concept of mapping. Coordinates in Euclidean space. The concept of mapping is of basic importance in differential geometry.

Let M and M' be two sets of points in three-dimensional Euclidean space R_3. (M or M' may contain all points of R_3 or only a subset of these points.) If a rule T is stated which associates a point P' of M' to every point P of M we say that a *mapping* or *transformation* (more exactly: *point transformation*) of the set M *into* the set M' is given. P' is called the *image point* of P, and P is called an *inverse image point* of P'. The set of the image points of all points of M is called the *image* of M. If every point of M' is an image point of at least one point of M the mapping is called a mapping of M *onto* M'.

A mapping T of M onto M' is called *one-to-one* if the image points of any pair of different points of M are different points of M'. Then there exists the *inverse mapping* of T, denoted by T^{-1}, which maps M' onto M such that every point P' of M' is mapped onto that point P of M which corresponds to P' with respect to the mapping T.

The set of all points whose distance from a point P is smaller than a positive number η is called a *neighbourhood* of P. Consequently this neighbourhood consists of all points in the interior of a sphere of radius η with centre at P. There are arbitrarily many different neighbourhoods of P each of which corresponds to a certain value of η. A mapping of a set M into a set M'

is said to be *continuous at a point* P of M if, for every neighbourhood U' of the image P' of P there exists a neighbourhood U of P whose image is contained in U'. The mapping is said to be *continuous* if it is continuous at every point of M. A one-to-one continuous mapping whose inverse mapping is also continuous is called a *topological mapping*. Point sets which can be topologically mapped onto each other are said to be *homeomorphic*.

A mapping is called a *rigid motion* if any pair of image points has the same distance as the corresponding pair of inverse image points.

We will now discuss some basic facts of the analytic geometry of the three-dimensional Euclidean space R_3 which we will need in our later investigations.

We first introduce a right-handed system of orthogonal parallel coordinates x_1, x_2, x_3 whose unit points on the axes, that is, the points with coordinates (1, 0, 0), (0, 1, 0), (0, 0, 1), respectively, have the distance 1 from the origin, cf. Fig. 1. Such a special right-handed system will be called a *Cartesian coordinate system*.

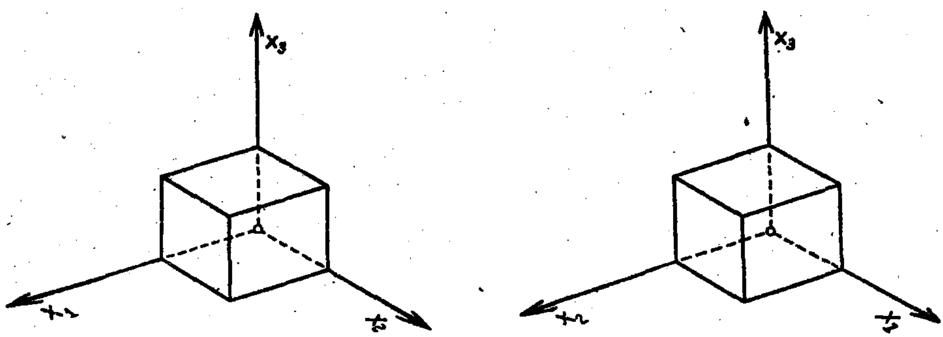

FIG. 1. Right-handed system of orthogonal parallel coordinates

FIG. 2. Left-handed system of orthogonal parallel coordinates

In general, a coordinate system is called *right-handed* if the axes, in their natural order, assume the same sort of orientation as the thumb, index finger, and middle finger of the right hand. A system is said to be *left-handed* if the axes, in their natural order, assume the same sort of orientation as the thumb, index finger, and middle finger of the left hand, cf. Fig. 2.

The notation x_1, x_2, x_3 for the coordinates is more convenient than the familiar x, y, z, for it enables us to use the abbreviated form (x_i) for the coordinates x_1, x_2, x_3 of a point.

Any other Cartesian coordinate system $\bar{x}_1, \bar{x}_2, \bar{x}_3$ is related to the given one by a special linear transformation of the form

$$\bar{x}_i = \sum_{k=1}^{3} a_{ik} x_k + b_i, \qquad (i = 1, 2, 3) \tag{3.1 a}$$

whose coefficients satisfy the conditions

$$\sum_{i=1}^{3} a_{ik} a_{il} = \delta_{kl} = \begin{cases} 0 & (k \neq l) \\ 1 & (k = l), \end{cases} \qquad (k, l = 1, 2, 3), \tag{3.1 b}$$

and

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 1. \tag{3.1 c}$$

The quantity δ_{kl} is called the *Kronecker symbol.*

The transition from one Cartesian coordinate system to another can be effected by a certain rigid motion of the axes of the original system. Such a motion is composed of a suitable translation and a suitable rotation. A rigid motion which carries a Cartesian coordinate system into another Cartesian coordinate system is called a *direct congruent transformation* (or *displacement*). We will now investigate (3.1) in somewhat greater detail.

Let m and n be natural numbers. A system of $m \cdot n$ quantities arranged in a rectangular array of m horizontal rows and n vertical columns is called a *matrix.* The quantities are called *elements* of that matrix. If m equals n the matrix is said to be *square,* and the number n is called the *order* of the matrix.

The coefficients a_{ik} figuring in (3.1) form a quadratic matrix

$$A = (a_{ik}) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

The corresponding determinant (3.1 c) will be denoted by

$$\det A = \det(a_{ik}).$$

If in particular A equals the '*unit matrix*'

$$(\delta_{ik}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

then (3.1 a) is of the form

$$\bar{x}_i = x_i + b_i, \qquad (i = 1, 2, 3). \tag{3.2}$$

This transformation corresponds to a translation of the coordinate system. If, moreover, $b_i = 0$ $(i = 1, 2, 3)$, we obtain from (3.2)

$$\bar{x}_i = x_i, \qquad (i = 1, 2, 3), \tag{3.3}$$

i.e. the transformed coordinates are the same as the original ones. Such a special transformation is called *the identical transformation.*

If $b_i = 0$ $(i = 1, 2, 3)$, and the coefficients a_{ik} are arbitrary but such that the conditions (3.1 b) and (3.1 c) are satisfied, then (3.1 a) corresponds to a rotation of the coordinate system with the origin as centre. A rotation is also called a *direct orthogonal transformation.*

We note that a transformation of the form

$$\bar{x}_i = \sum_{k=1}^{3} a_{ik} x_k, \qquad \sum_{i=1}^{3} a_{ik} a_{il} = \delta_{kl}, \qquad \det(a_{ik}) = -1,$$

can be geometrically interpreted as a motion composed of a rotation about the origin and a reflection in a plane. A transformation of this type is called an *opposite orthogonal transformation.* It transforms a right-handed coordinate system into a left-handed one and vice versa. An example of a reflection in a plane (in this particular case, in the $x_2 x_3$-coordinate plane) is given by the transformation $\bar{x}_1 = -x_1$, $\bar{x}_2 = x_2$, $\bar{x}_3 = x_3$. Direct and opposite orthogonal transformations are called *orthogonal transformations,* and the corresponding matrices are referred to as *orthogonal matrices.* A transformation which is composed of translations, rotations, and an odd number of reflections is called an *opposite congruent transformation.* We should note that every translation or rotation can be composed of two suitable reflections.

We will now point out that (3.1) can be interpreted in two different ways:

(*Alias.*) Formerly we interpreted (3.1) as a coordinate transformation; (x_i) and $(\bar{x}_i)$ are then the coordinates of one and the same point with respect to two different Cartesian coordinate systems.

(*Alibi.*) The relation (3.1) can also be interpreted as a mapping or point transformation. Then (x_i) and $(\bar{x}_i)$ represent the coordinates of two different points with respect to one and the same Cartesian coordinate system; that is, the coordinate system remains fixed and the location of the points is changed.

Both interpretations are closely related to each other. For, in consequence of the above remarks, the transition from one interpretation to the other can be effected in the following manner. Instead of imposing a direct congruent transformation on the given Cartesian coordinate system, one can just as well move the geometric configuration, that is, change its