Nonlinear Dynamics and Turbulence

Edited by

G I Barenblatt

G Iooss

D D Joseph

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Edited by

G I Barenblatt
P P Shirsov Institute of Oceanology
USSR Academy of Sciences

G Iooss
Institut de Mathématiques et Sciences Physiques
Université de Nice

D D Joseph
Department of Aerospace Engineering and Mechanics
University of Minnesota

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Foreword

The International Society for the Interaction of Mechanics and Mathematics (ISIMM) was founded in 1977. Its purpose is to promote cooperative research involving the fields of mechanics and pure mathematics.

Its Executive Committee decided that, from time to time, scholarly works relevant to the Society's interests should, by invitation, be published under its auspices. The present volume is one in this series which, it is hoped, will help to advance the objective of the Society.

The Editorial Board

List of Contributors

V S Afraimovich Research Institute of Applied Mathematics and Cybernetics, N I Lobachevsky Gorky State University

D G Aronson School of Mathematics, University of Minnesota

G I Barenblatt P P Shirsov Institute of Oceanology, USSR Academy of Sciences

Yu N Belayev Institute of Mechanics, Moscow State University

L A Bunimovich P P Shirsov Institute of Oceanology, USSR Academy of Sciences

F H Busse Department of Planetary and Space Science, University of California Los Angeles

A I Chernykh Institute of Automation and Electrometry, Siberian Branch, USSR Academy of Sciences

M A Chory The Analytic Sciences Corporation, Reading, Massachusetts

M J Feigenbaum Physics Department, Cornell University

C Foias Department of Mathematics, Indiana University

J P Gollub Haverford College and the University of Pennsylvania

B D Greenspan Department of Mathematics, Vanderbilt University

G R Hall School of Mathematics, University of Minnesota

P J Holmes Department of Theoretical and Applied Mechanics and Center for Applied Mathematics, Cornell University

W F Langford Department of Mathematics and Statistics, University of Guelph, Ontario

V S L'vov Institute of Automation and Electrometry, Siberian Branch, USSR Academy of Sciences

R P McGehee School of Mathematics, University of Minnesota

R V Plykin Moscow Institute of Physical Engineering

Y Pomeau Service de Physique Théorique, Commissariat à l'Énergie Atomique, France

A A Predtechensky Institute of Automation and Electrometry, Siberian Branch, USSR Academy of Sciences

George R Sell School of Mathematics, University of Minnesota

L P Shil'nikov Research Institute of Applied Mathematics and Cybernetics, N I Lobachevsky Gorky State University

F Takens Mathematisch Instituut, Rijksuniversiteit Groningen

R Temam Mathématique Université de Paris-Sud, Orsay

Y M Treve La Jolla Institute, California

I M Yavorskaya Space Research Institute, USSR Academy of Sciences

G M Zaslavsky L V Kirensky Institute of Physics, Siberian Branch, USSR Academy of Sciences

Preface

Turbulence, narrowly viewed, is a mode of motion of fluids. There are *at least* two modes of motion: a laminar one whose dynamics mimics the symmetries of imposed external conditions, and a turbulent one which is unsteady and chaotic even when the external conditions are perfectly steady and regular. Osborne Reynolds was the first to write down a systematic study of laminar and turbulent flow in pipes. He showed that the transition between laminar and turbulent flows depends on a single dimensionless number, now called the Reynolds number. For Reynolds numbers inferior to a critical one the (laminar) flow is as steady and regular as the imposed data (the pressure drop and boundary conditions), whereas in the other case the laminar flow loses stability to turbulent flow, which is unsteady and irregular even though the imposed data are steady and regular. Since all motions of fluids are governed by equations of motion, the existence of a connection between dynamics and turbulence is apparent. However, we cannot solve the equations in the turbulent case and are unable even to establish some well-known properties of the observed motions in a mathematically satisfying way.

Dynamics is a larger and more general subject than turbulence. It applies to all motions of all material bodies. The mathematical theory of dynamics seeks properties of solutions of the equations of motion. It has been formulated and studied in abstract form as a topic in the branch of mathematics known as dynamical systems. As is frequently the case, the subject was simplified in some sense by looking at it in a more general context. In this context it was learned that deterministic problems, governed only by some few benign nonlinear ordinary differential equations or by iterations of maps in finite-dimensional spaces, could give rise to chaotic turbulent-like behaviour, highly sensitive to initial conditions and unpredictable after long times or many iterations.

The fact that dynamical systems with a few degrees of freedom can possess stochasticity due to strong (exponential repulsivity) sensitivity to initial conditions has been known for many years. E. N. Lorenz, in 1963, was the first to show that a deterministic problem of fluid mechanics could lead to turbulent-like behaviour with chaotic dynamics. Lorenz considered the problem of thermal convection governed by Navier–Stokes-type partial differential (Oberbeck–Boussincsq) equations. He supposed that the solutions of these equations could be expressed by a Galerkin approximation which he truncated drastically. In this way he came to three ordinary differential equations with bilinear nonlinearities.

Lorenz studied the solution trajectories numerically and interpreted the results in a mathematically precise way, using the qualitative theory of differential equations. He showed numerically that solutions were attracted to a certain special set in the three-dimensional phase space. This subset is more complicated than a two-dimensional surface and the trajectories are sensitive to changes in initial conditions: they have a very complicated structure and they give rise to continuous spectra. Sensitivity to initial conditions means that very slight changes in initial conditions give rise eventually to entirely different trajectories. The system has the property of mixing. This means that typical trajectories essentially 'forget' their initial conditions so that autocorrelations computed on functionals of the solutions decay eventually to zero: that is, two events, widely separated, are uncorrelated so that it is impossible to predict one event from another if the intervening time is large. Turbulence in fluids has this property. A continuous frequency spectrum is associated with mixing flows, because a discrete spectrum implies some kind of (almost) periodicity for which the autocorrelation between the same quantity at different times need not decrease to zero as the time lapse between events approaches infinity.

Lorenz's paper attracted attention some years after the appearance in 1972 of a paper by Ruelle and Takens which used the theory of topological dynamics to introduce the idea that turbulence in fluids would appear after a few bifurcations. This notion ran against the then popular theory which had been introduced earlier by Landau (1944) and Hopf (1948). The ideas of Landau and Hopf had a natural evolution. In the earlier days of the study of hydrodynamic problems it was thought that the conditions for the development of turbulence could be ascertained from the theory of stability of laminar flows. This study of stability showed that laminar flows are indeed unstable but that the flow which replaces them is also a laminar one, with a different symmetry pattern in either space or time; for example, a time-periodic motion may replace a steady motion after the loss of stability. It was easy to imagine that the loss of the stability of the periodic motion could lead to a flow with two frequencies, and so on, to n frequencies, as the Reynolds number was increased. So their theory was that, when transition is progressive, at any stage there exists in the phase space of solutions corresponding to the n frequencies an n-dimensional torus which is invariant and attracting.

The ideas of Landau and Hopf do not give a good description of transition to turbulence because quasiperiodic solutions are not mixing and do not exhibit a continuous spectrum, decaying autocorrelation or high sensitivity to initial conditions. Ruelle and Takens noted that it was possible for attracting sets having the stochasticity (mixing, etc.) required for turbulence to arise after a few bifurcations of the Landau type. A small perturbation of a quasiperiodic flow on a three-dimensional torus can lead to a strange attractor. In fact, the structure of bifurcations changes drastically after a few elementary bifurcations of the Landau–Hopf type. For instance, we have already noted that quasiperiodic flows can undergo transition to chaotic behaviour. Another possible transition is by means of a cascade of period-doubling bifurcations of the Feigenbaum type.

Mathematical studies require good definitions on which theories can be built. Turbulence in fluids has never had so precise a mathematical definition, though the general notion of turbulence in dynamical systems can be made precise. It remains, of course, to show that this general notion is actually useful in making theories to explain what we observe as turbulence in fluid flows. Many of the papers in this volume have as their aim the elucidation of the implications of the generalized concept of turbulence and the study of its utility in applications to observed turbulence in fluids. To achieve such an aim it is necessary to combine analytical, computer and experimental studies. The editors have tried here to represent all such approaches.

It will be clear to readers of this volume that the new results coming from the theory of nonlinear dynamics fit very well some simple systems of small dimension—for example, convection in small boxes, flow between rotating cylinders and rotating spheres. The dynamics of such problems are governed by partial differential equations of the Navier–Stokes type but are actually controlled by finite-dimensional ordinary differential equations of a type that can be obtained by appropriate Galerkin approximations. So the dynamics is actually set in a finite-dimensional phase space even though the governing problem is an infinite-dimensional one. When the configuration of the experiments and the Reynolds number are appropriately chosen, it appears that the number of relevant dimensions is small and there is encouraging agreement between the new theories and experiments. The study of the number of finite dimensions necessary to represent the qualitative features of the true dynamics is a new and important one, which is less well understood when the number of finite dimensions is large. The classical cases of turbulence in jets and in pipe and boundary layer flows are among those which presumably require a large (if finite) number of dimensions. It remains to be seen if these classical cases, and other practical problems of turbulence in fluids, can be usefully treated by these new ideas from the theory of nonlinear dynamics.

The editors and authors hope that this collection of papers will help readers to understand what has been achieved, and to achieve more. We also believe that this volume reflects the best efforts of our time, by scientists of various countries, to understand the nonlinear dynamics governing turbulence.

G. I. B.
G. I.
D. D. J.

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1 Strange Attractors and Quasiattractors

V S Afraimovich and L P Shil'nikov

1.1 Introduction

The attention of specialists in various fields has frequently been drawn in recent years to certain special dynamical models with a complicated behaviour of trajectories. This relates primarily to an attempt to answer the following fundamental question: can stochastic oscillations arise in systems of markedly pronounced dynamical character? In this case the investigation of hydrodynamic models becomes especially important in order to explain the turbulence phenomenon.

The problem of the onset of turbulence and the description of developed turbulence are basic problems in hydrodynamics. Whereas the approach to the second problem is based on ideas and methods of the probability theory (see Monin and Yaglom [1]), the modern approach to the problem of turbulence onset relies upon the idea that a dynamical system, i.e., the actual Navier–Stokes equations or a finite-dimensional system which is their Galerkin approximation, can serve as a model for fluid motion. Hence the methods of quantitative theory of dynamical systems can be applied to this system. And if this is the case, a question arises: what is the mathematical prototype of turbulent motion? (Let us note that in a similar situation the answer to the question about the mathematical character of self-oscillations was given in the 1930s by A. A. Andronov on the basis of the Poincaré–Lyapunoff theory.)

Attempts to answer this question have been made for decades. For instance, in the 1940s Landau and Hopf suggested that an asymptotically stable torus with a quasiperiodic trajectory can be an adequate mathematical prototype for turbulent motion (at that time limiting sets consisting of almost periodic motions seemed to be the most 'complicated' ones). As for stochastic motions, objects that are the mathematical prototypes for them are more complicated than periodic motions or the motions on a torus with a quasiperiodic trajectory. These are the so-called strange attractors (see, e.g., [2]), in which the stability of the set as a whole is combined with instability of the individual trajectories. Mathematicians are well familiar with hyperbolic strange attractors such as submanifolds with Y-structure [3], Smale–Williams's solenoids [4, 5], R. V. Plykin's solenoids [6], etc. Since the methods of ergodic theory, as applied to hyperbolic sets, have been developed in some depth (see Chapter 5 by Bunimovich), it would seem that an apparatus now exists for the study of systems with a complicated behaviour of trajectories. It is this approach which is used by Ruelle and Takens [7], who propose Smale–Williams's solenoids as a mathematical prototype for turbulent flow.

One has to admit, however, that the possibility of the appearance of hyperbolic attractors in simple models still remains problematic. Among nonstructurally stable strange attractors, those are of primary importance in which the following characteristic properties of hyperbolic sets are present: transitivity, an everywhere density of saddle periodic motions, the existence of a homoclinic curve for each periodic motion (for definitions see below). A strange attractor will be called quasihyperbolic if it possesses the above-mentioned properties and preserves them under small perturbations and if, at the same time, it can be nonstructurally stable. The existence of quasihyperbolic attractors has been proved by Afraimovich *et al.* [8]. Numerical analyses performed by Lorenz [9], Afraimovich *et al.* [8] and others (see, e.g., [10, 11]) show that quasihyperbolic attractors may exist in simple three-dimensional models, and above all in Lorenz's triplet:

$$\dot{x} = -\sigma(x-y), \qquad \dot{y} = -xz + rx - y, \qquad \dot{z} = xy - bz. \tag{1.1}$$

It should be noted that Afraimovich *et al.* have formulated analytical conditions for Poincaré mapping; if these conditions are satisfied, a quasihyperbolic attractor does exist (see below). An appropriate 'transversal', namely the plane $z = r - 1$, does exist for system (1.1). However, there is almost no information concerning the analytical properties of the Poincaré mapping of this plane along the trajectories of system (1.1). That is why Afraimovich *et al.* verify the theoretical conditions for Lorenz's model numerically.

Let us also present a model for the excitation of oscillations in a laser [12]:

$$\dot{x} = -\gamma_3(x-y), \qquad \dot{y} = \gamma_2(xz-y), \qquad \dot{z} = -\gamma_1(z-z_0) + \gamma_1(z_0-1)xy \tag{1.2}$$

and a model for a simple 'dynamo' [13]:

$$\frac{dw}{dt} = R - zy - \nu w, \qquad \frac{dz}{dt} = wy - z, \qquad \frac{dy}{dt} = \gamma(z-y). \tag{1.3}$$

The specific feature of these models is that their mathematical descriptions are similar in that they can be reduced to Eq. (1.1) by a linear change of variables. The reason for this 'similarity' appears to lie in the important fact that there is usually present a stable limiting set that causes stochastic behaviour of trajectories in three-dimensional systems (a quasihyperbolic attractor without lacunae—see below for an exact definition), which we shall call a Lorenz attractor; this therefore requires detailed investigation. The major part of this chapter (Sections 1.2–1.4) is devoted to quasihyperbolic attractors.

In special cases, however, it is frequently difficult to state whether a system is hyperbolic or quasihyperbolic on an attracting set, though certain easily verified consequences of hyperbolicity are valid: strong dependence on the initial conditions; irregular, chaotic behaviour of the trajectory; etc., Moreover, the following phenomenon has recently been discovered: in certain cases, where the existence of stable periodic motions together with a nontrivial hyperbolic subset (in an invariant region), to which the trajectory appears to be attracted, is proved theoretically, these motions cannot be 'picked up' either by computer or by other simulation methods. However, the observer is dealing with a stochastic motion. We propose the term 'quasiattractor' to denote such situations. We shall say, to be more exact, that an attracting set is an ε-quasiattractor if (i) it contains a nontrivial hyperbolic subset and (ii) it possesses no equilibrium state or stable periodic motion whose period would be less than $1/\varepsilon$. In cases where the value of ε is not specified, such a limiting set will be called a quasiattractor.

Numerical calculations show that in the Lorenz model (Eq. (1.1)) a quasiattractor exists for values of r from 28 to $\simeq 100$, though, as was proved by Afraimovich *et al.* [14], stable periodic motions may also exist. The reason for their appearance may be twofold:

(1) destruction of the saddle-focus separatrix loop whose existence, in turn, follows from the existence of a contour formed by the saddle, saddle-focus, and trajectories which connect them [15, 16];
(2) stable periodic motions. Stable periodic motions arise as a result of bifurcations taking place in systems close to systems with a nonstructurally stable homoclinic curve, the saddle value of periodic motion having a homoclinic curve being less than 1, which is determined by the fact that the divergence of the vector field (1.1) is negative (see [17, 18]).

Other cases where quasiattractors arise were considered by Afraimovich and Shil'nikov ([19–21]); they are connected with the action of small periodic perturbations on a self-oscillating system with weak interaction between two self-oscillating systems, and with the disappearance of nonstructurally stable motion of the saddle-node type. Mathematical effects that arise in such problems have partially been considered in [22–24]. In the above-mentioned cases a quasiattractor arises as a result of the destruction of a stable two-dimensional torus, and from the mathematical standpoint the problem is reduced to the mapping of a ring into itself. In Section 1.5 a mechanism is considered for the appearance of an attractor for the simplest mapping of a plane domain. These mappings,