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实定理的复证明

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图字：01-2016-2496 号

实定理的复证明
Shidingli de Fuzhengming

图书在版编目 (CIP) 数据

实定理的复证明 = Complex Proofs of Real Theorems : 英文 / (美) 彼得 · D · 拉克斯 (Peter D. Lax) , (美) 劳伦斯 · 查克曼 (Lawrence Zalcman) 著 . — 影印本 . — 北京 : 高等教育出版社 , 2017.4
ISBN 978-7-04-047000-0
Ⅰ. ①实… Ⅱ. ①彼… ②劳… Ⅲ. ①定理证明 — 英文 Ⅳ. ①O141
中国版本图书馆 CIP 数据核字 (2016) 第 326755 号

策划编辑 李 鹏　　责任编辑 李 鹏
封面设计 张申申　　责任印制 毛斯璐

出版发行 高等教育出版社
社址 北京市西城区德外大街 4 号
邮政编码 100120
购书热线 010-58581118
咨询电话 400-810-0598
网址 http://www.hep.edu.cn
http://www.hep.com.cn
网上订购 http://www.hepmall.com.cn
http://www.hepmall.com
http://www.hepmall.cn
印刷:北京新华印刷有限公司

开本 787mm × 1092mm 1/16
印张 6.75
字数 170 千字
版次 2017 年 4 月第 1 版
印次 2017 年 4 月第 1 次印刷
定价 55.00 元

[物 料 号 47000-00]

美国数学会经典影印系列

出版者的话

近年来，我国的科学技术取得了长足进步，特别是在数学等自然科学基础领域不断涌现出一流的研究成果。与此同时，国内的科研队伍与国外的交流合作也越来越密切，越来越多的科研工作者可以熟练地阅读英文文献，并在国际顶级期刊发表英文学术文章，在国外出版社出版英文学术著作。

然而，在国内阅读海外原版英文图书仍不是非常便捷。一方面，这些原版图书主要集中在科技、教育比较发达的大中城市的大型综合图书馆以及科研院所的资料室中，普通读者借阅不甚容易；另一方面，原版书价格昂贵，动辄上百美元，购买也很不方便。这极大地限制了科技工作者对于国外先进科学技术知识的获取，间接阻碍了我国科技的发展。

高等教育出版社本着植根教育、弘扬学术的宗旨服务我国广大科技和教育工作者，同美国数学会（American Mathematical Society）合作，在征求海内外众多专家学者意见的基础上，精选该学会近年出版的数十种专业著作，组织出版了“美国数学会经典影印系列”丛书。美国数学会创建于 1888 年，是国际上极具影响力的专业学术组织，目前拥有近 30000 会员和 580 余个机构成员，出版图书 3500 多种，冯·诺依曼、莱夫谢茨、陶哲轩等世界级数学大家都是其作者。本影印系列涵盖了代数、几何、分析、方程、拓扑、概率、动力系统等所有主要数学分支以及新近发展的数学主题。

我们希望这套书的出版，能够对国内的科研工作者、教育工作者以及青年学生起到重要的学术引领作用，也希望今后能有更多的海外优秀英文著作被介绍到中国。

高等教育出版社

2016 年 12 月

To our wives,
Lori and Adrienne

Preface

At the middle of the twentieth century, the theory of analytic functions of a complex variable occupied an honored, even privileged, position within the canon of core mathematics. This "particularly rich and harmonious theory," averred Hermann Weyl, "is the showpiece of classical nineteenth century analysis."[1] Lest this be mistaken for a gentle hint that the subject was getting old-fashioned, we should recall Weyl's characterization just a few years earlier of Nevanlinna's theory of value distribution for meromorphic functions as "one of the few great mathematical events in our century."[2] Leading researchers in areas far removed from function theory seemingly vied with one another in affirming the "permanent value"[3] of the theory. Thus, Clifford Truesdell declared that "conformal maps and analytic functions will stay current in our culture as long as it lasts";[4] and Eugene Wigner, referring to "the many beautiful theorems in the theory ... of power series and of analytic functions in general," described them as the "most beautiful accomplishments of [the mathematician's] genius."[5] Little wonder, then, that complex function theory was a mainstay of the graduate curriculum, a necessary and integral part of the common culture of all mathematicians.

Much has changed in the past half century, not all of it for the better. From its central position in the curriculum, complex analysis has been pushed to the margins. It is now entirely possible at some institutions to obtain a Ph.D. in mathematics without being exposed to the basic facts of function theory, and (incredible as it may seem) even students specializing in analysis often fulfill degree requirements by taking only a single semester of complex analysis. This, despite the fact that complex variables offers the analyst such indispensable tools as power series, analytic continuation, and the Cauchy integral. Moreover, many important results in real analysis use complex variables in their proofs. Indeed, as Painlevé wrote already at the end of the nineteenth century, "Between two truths of the real domain, the easiest and shortest path quite often passes through the complex

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[1]Hermann Weyl, *A half-century of mathematics*, Amer. Math. Monthly **58** (1951), 523-553, p. 526.

[2]Hermann Weyl, *Meromorphic Functions and Analytic Curves*, Princeton University Press, 1943, p. 8.

[3]G. Kreisel, *On the kind of data needed for a theory of proofs*, Logic Colloquium **76**, North Holland, 1977, pp. 111-128, p. 118.

[4]C. Truesdell, *Six Lectures on Modern Natural Philosophy*, Springer-Verlag, 1966, p. 107.

[5]Eugene P. Wigner, *The unreasonable effectiveness of mathematics in the natural sciences*, Comm. Pure Appl. Math. **13** (1960), 1-14, p. 3.

domain,"[6] a claim endorsed and popularized by Hadamard.[7] Our aim in this little book is to illustrate this thesis by bringing together in one volume a variety of mathematical results whose formulations lie outside complex analysis but whose proofs employ the theory of analytic functions. The most famous such example is, of course, the Prime Number Theorem; but, as we show, there are many other examples as well, some of them basic results.

For whom, then, is this book intended? First of all, for everyone who loves analysis and enjoys reading pretty proofs. The technical level is relatively modest. We assume familiarity with basic functional analysis and some elementary facts about the Fourier transform, as presented, for instance, in the first author's *Functional Analysis* (Wiley-Interscience, 2002), referred to henceforth as [**FA**]. In those few instances where we have made use of results not generally covered in the standard first course in complex variables, we have stated them carefully and proved them in appendices. Thus the material should be accessible to graduate students. A second audience consists of instructors of complex variable courses interested in enriching their lectures with examples which use the theory to solve problems drawn from outside the field.

Here is a brief summary of the material covered in this volume. We begin with a short account of how complex variables yields quick and efficient solutions of two problems which were of great interest in the seventeenth and eighteenth centuries, viz., the evaluation of $\sum_1^\infty 1/n^2$ and related sums and the proof that every algebraic equation in a single variable (with real or even complex coefficients) is solvable in the field of complex numbers. Next, we discuss two representative applications of complex analysis to approximation theory in the real domain: weighted polynomial approximation on the line and uniform approximation on the unit interval by linear combinations of the functions $\{x^{n_k}\}$, where $n_k \to \infty$ (Müntz's Theorem). We then turn to applications of complex variables to operator theory and harmonic analysis. These chapters form the heart of the book. A first application to operator theory is Rosenblum's elegant proof of the Fuglede-Putnam Theorem. We then discuss Toeplitz operators and their inversion, Beurling's characterization of the invariant subspaces of the unilateral shift on the Hardy space H^2 and the consequent divisibility theory for the algebra $\mathcal{B}$ of bounded analytic functions on the disk or half-plane, and a celebrated problem in prediction theory (Szegő's Theorem). We also prove the Riesz-Thorin Convexity Theorem and use it to deduce the boundedness of the Hilbert transform on $L^p(\mathbb{R})$, $1 < p < \infty$. The chapter on applications to harmonic analysis begins with D.J. Newman's striking proof of Fourier uniqueness via complex variables; continues on to a discussion of a curious functional equation and questions of uniqueness (and nonuniqueness) for the Radon transform; and then turns to the Paley-Wiener Theorem, which together with the divisibility theory for $\mathcal{B}$ referred to above is exploited to provide a simple proof of the Titchmarsh Convolution Theorem. This chapter concludes with Hardy's Theorem, which quantifies the fact that a function and its Fourier transform cannot both tend to zero

[6] "Entre deux vérités du domain réel, le chemin le plus facile et le plus court passe bien souvent par le domaine complexe." Paul Painlevé, *Analyse des travaux scientifiques*, Gauthier-Villars, 1900, pp.1-2.

[7] "It has been written that the shortest and best way between two truths of the real domain often passes through the imaginary one." Jacques Hadamard, *An Essay on the Psychology of Invention in the Mathematical Field*, Princeton University Press, 1945, p. 123.

too rapidly. The final chapters are devoted to the Gleason-Kahane-Żelazko Theorem (in a unital Banach algebra, a subspace of codimension 1 which contains no invertible elements is a maximal ideal) and the Fatou-Julia-Baker Theorem (the Julia set of a rational function of degree at least 2 or a nonlinear entire function is the closure of the repelling periodic points). We end on a high note, with a proof of the Prime Number Theorem. A coda deals very briefly with two unusual applications: one to fluid dynamics (the design of shockless airfoils for partly supersonic flows), and the other to statistical mechanics (the stochastic Loewner evolution).

To a certain extent, the choice of topics is canonical; but, inevitably, it has also been influenced by our own research interests. Some of the material has been adapted from [**FA**]. Our title echoes that of a paper by the second author.[8]

Although this book has been in the planning stages for some time, the actual writing was done during the Spring and Summer of 2010, while the second author was on sabbatical from Bar-Ilan University. He thanks the Courant Institute of Mathematical Sciences of New York University for its hospitality during part of this period and acknowledges the support of Israel Science Foundation Grant 395/07.

Finally, it is a pleasure to acknowledge valuable input from a number of friends and colleagues. Charles Horowitz read the initial draft and made many useful comments. David Armitage, Walter Bergweiler, Alex Eremenko, Aimo Hinkkanen, and Tony O'Farrell all offered perceptive remarks and helpful advice on subsequent versions. Special thanks to Miriam Beller for her expert preparation of the manuscript.

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[8]Lawrence Zalcman, *Real proofs of complex theorems (and vice versa)*, Amer. Math. Monthly **81** (1974), 115-137.

Contents

CHAPTER 1

Early Triumphs

Nothing illustrates the extraordinary power of complex function theory better than the ease and elegance with which it yields results which challenged and often baffled the very greatest mathematicians of an earlier age. In this brief chapter, we consider two such examples: the solution of the "Basel Problem" of evaluating $\sum_1^\infty 1/n^2$ and the proof of the Fundamental Theorem of Algebra. To be sure, these achievements predate the development of the theory of analytic functions; but, even today, complex variables offers the simplest and most transparent approach to these beautiful results.

1.1. The Basel Problem

Surely one of the most spectacular applications of complex variables is the use of Cauchy's Theorem and the Residue Theorem to find closed form expressions for definite integrals and infinite sums. As an illustration, we evaluate the sums

$$\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}}, \qquad k = 1, 2, \dots .$$

The function

$$H(z) = \frac{2\pi i}{e^{2\pi i z} - 1}$$

is meromorphic on $\mathbb{C}$ with simple poles at the integers, each having residue 1, and no other singularities in the finite plane. It follows that if f is a function analytic near the point $z = n$ ($n \in \mathbb{Z}$), then $\operatorname{Res}(H(z)f(z), n) = f(n)$. We choose $f(z) = 1/z^{2k}$ for k fixed and consider the integral

$$I_N = \frac{1}{2\pi i} \int_{\Gamma_N} H(z) \frac{1}{z^{2k}} dz, \tag{1.1}$$

where N is a positive integer and Γ_N is the positively oriented boundary of the square with vertices at the points $(N + 1/2)(\pm 1 \pm i)$. By the Residue Theorem,

$$I_N = \sum_{n=-N}^{N} \operatorname{Res}\left(H(z)\frac{1}{z^{2k}}, n\right) = \operatorname{Res}\left(H(z)\frac{1}{z^{2k}}, 0\right) + 2\sum_{n=1}^{N} \frac{1}{n^{2k}}. \tag{1.2}$$

A routine estimate shows that H is uniformly bounded on Γ_N with bound independent of N. Thus

$$H(z)\frac{1}{z^{2k}} = O\left(\frac{1}{N^{2k}}\right) \qquad \text{on} \qquad \Gamma_N;$$

and since Γ_N has length $8N + 4$, it follows from (1.1) that

$$I_N = O\left(\frac{1}{N^{2k-1}}\right).$$

Thus $\lim_{N\to\infty} I_N = 0$, so from (1.2), we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = -\frac{1}{2} \operatorname{Res}\left(H(z)\frac{1}{z^{2k}}, 0\right). \tag{1.3}$$

To evaluate the right hand side of (1.3) explicitly, recall that the Bernoulli numbers B_n are defined by

$$\frac{x}{e^x - 1} = \sum_{\ell=0}^{\infty} \frac{B_\ell x^\ell}{\ell!}. \tag{1.4}$$

In particular, $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$, $B_8 = -1/30$, $B_{10} = 5/66$, $B_{12} = -691/2730$. Now from (1.4), we have

$$H(z) = \frac{2\pi i}{e^{2\pi i z} - 1} = \sum_{\ell=0}^{\infty} \frac{B_\ell (2\pi i)^\ell z^{\ell-1}}{\ell!},$$

so that the coefficient of $1/z$ in the Laurent expansion of $H(z)/z^{2k}$ about 0 is given by

$$\operatorname{Res}\left(H(z)\frac{1}{z^{2k}}, 0\right) = \frac{(-1)^k B_{2k}(2\pi)^{2k}}{(2k)!}.$$

Plugging this into (1.3) yields

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k+1} 2^{2k-1}}{(2k)!} B_{2k} \pi^{2k},$$

which is the desired formula. In particular, taking $k = 1$, we have

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

COMMENTS. 1. Evaluating the sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$ was a celebrated problem in the mathematics of the late seventeenth and early eighteenth centuries. Originally posed by Pietro Mengoli in 1644, it was brought to public attention by Jacob Bernoulli in his *Tractatus de Seriebus Infinitis* (1689) and became known as the Basel Problem. After many unsuccessful attempts by leading mathematicians, it was finally solved in 1735 by Leonhard Euler, who produced a rigorous proof of the result in 1741. Euler went on to discover the general formula for $\zeta(2k)$, evaluating the sums explicitly for k up to 13. Of course, Euler's arguments did not make use of complex analysis, as that subject did not yet exist.

2. Expressing $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$ in a simple closed form (or proving that no such expression exists) remains an open problem of considerable interest; ditto for higher odd powers. It is known (Apéry) that $\zeta(3)$ is an irrational number; for a proof, see [**B**].

3. An extensive array of applications of the calculus of residues are displayed in the two volumes [**MK1**], [**MK2**].

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1.2. The Fundamental Theorem of Algebra

The Fundamental Theorem of Algebra (FTA) asserts that a nonconstant polynomial

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 \tag{1.5}$$

with complex coefficients must vanish somewhere in the complex plane. Eighteenth century attempts to establish this result (for polynomials with real coefficients) by such worthies as Euler, Lagrange, and Laplace all proved fatally flawed; and even the geometric proof proposed by Gauss in 1799 had a (topological) gap, which was filled only in 1920 (by Alexander Ostrowski [**O**]; cf. [**Sm**, pp. 4-5]). Thus, the first rigorous proof of the theorem, published by Argand in 1814, marks an early high water mark for nineteenth century mathematics.

Complex function theory offers a particularly efficient approach for proving FTA; and proofs using such results as Liouville's Theorem, the Maximum Principle, the Argument Principle, and Rouché's Theorem appear in the standard texts. Surprisingly, however, the simplest and shortest proof, based on the Cauchy Integral Formula for circles, does not seem to have been recorded in the textbook literature.

Proof of FTA. Let the polynomial p be given by (1.5), where $n \geq 1$ and $a_n \neq 0$. First observe that

$$\lim_{R\to\infty} |p(Re^{i\theta})| = \infty \qquad \text{uniformly in } \theta \tag{1.6}$$

since

$$|p(z)| \geq |z|^n(|a_n| - |a_{n-1}|/|z| - \cdots - |a_0|/|z|^n) > \frac{|a_n|}{2}|z|^n$$

for z sufficiently large.

Now suppose that p does not vanish on $\mathbb{C}$. Then $q = 1/p$ is analytic throughout $\mathbb{C}$ and $q(0) = 1/p(0) \neq 0$. By Cauchy's integral formula,

$$q(0) = \frac{1}{2\pi i}\int_{|z|=R} \frac{q(z)}{z}dz = \frac{1}{2\pi}\int_0^{2\pi} q(Re^{i\theta})d\theta \tag{1.7}$$

for all $R > 0$. But the integral on the right hand side of (1.7) tends to 0 by (1.6) as $R \to \infty$, and we have the desired contradiction. □

Comment. The proof given above is taken from [**Z**]; cf. [**Sc**] and the discussion in [**V**].

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CHAPTER 2

Approximation

Analyticity can often be exploited to advantage in the study of problems of approximation, even when the objects to be approximated are functions of a real variable. We illustrate this point in the following two sections. In each of them, an essential role is played by the following basic result from functional analysis, known as the *spanning criterion.*

SPANNING CRITERION. A point z of a normed linear space X belongs to the closed linear span Y of a subset $\{y_j\}$ of X if and only if every bounded linear functional ℓ that vanishes on the subset vanishes at z, that is,

$$(*) \qquad \ell(y_j) = 0 \qquad \text{for all} \qquad y_j$$

implies that $\ell(z) = 0$.

In particular, the linear combinations of $\{y_j\}$ span all of X if and only if no bounded linear functional ℓ satisfies (*) other than $\ell = 0$.

For the proof, based on the Hahn-Banach Theorem, see [**FA**, pp. 77-78].

2.1. Completeness of Weighted Powers

Let w be a given positive continuous function defined on $\mathbb{R}$ that decays exponentially as $|t| \to \infty$:

$$(2.1) \qquad 0 < w(t) < ae^{-c|t|}, \qquad c > 0.$$

Denote by C_0 the set of continuous functions on $\mathbb{R}$ that vanish at ∞ :

$$\lim_{|t| \to \infty} x(t) = 0.$$

Then C_0 is a Banach space under the maximum norm.

THEOREM 2.1. *The functions $t^n w(t)$, $n = 0, 1, 2, \ldots$, belong to C_0; their closed linear span is all of C_0. That is, every function in C_0 can be approximated uniformly on $\mathbb{R}$ by weighted polynomials.*

PROOF. We apply the spanning criterion. Let ℓ be any bounded linear functional over C_0 that vanishes on the functions $t^n w$:

$$(2.2) \qquad \ell(t^n w) = 0, \qquad n = 0, 1, \ldots .$$

Let z be a complex variable, $|\operatorname{Im} z| < c$. Then $w(t)e^{izt}$ belongs to C_0, and so

$$f(z) = \ell(we^{izt})$$

is defined in the strip $|\operatorname{Im} z| < c$. We claim that f is analytic there. For the complex difference quotients of we^{izt} tend to $iwte^{izt}$ in the norm of C_0, and so

$$f'(z) = \lim_{\delta\to 0} \frac{f(z+\delta) - f(z)}{\delta} = \lim_{\delta\to 0} \ell\left(w\frac{e^{i(z+\delta)t} - e^{izt}}{\delta}\right) = \ell(iwte^{izt}).$$

Similarly for the higher derivatives; in particular, using (2.2), we have

$$\left.\frac{d^n f}{dz^n}\right|_{z=0} = i^n \ell(wt^n) = 0, \qquad n = 0, 1, \dots .$$

Since f is analytic, the vanishing of all its derivatives at $z = 0$ means that $f(z) \equiv 0$ in the strip; in particular,

$$f(z) = \ell(we^{izt}) = 0 \qquad \text{for all } z \text{ real.}$$

By the spanning criterion, it follows that all functions we^{izt} belong to the closed linear span of $t^n w$.

According to the Weierstrass approximation theorem, every continuous *periodic* function h is the uniform limit of trigonometric polynomials. It follows that wh belongs to the closed linear span of the functions we^{izt}, z real, hence of the functions $t^n w$. Let y be any continuous function of compact support; define x by

$$x = \frac{y}{w}. \tag{2.3}$$

Denote by h a $2p$ periodic function such that

$$x(t) \equiv h(t) \qquad \text{for } |t| < p, \tag{2.4}$$

where p is chosen so large that the support of x is contained in the interval $|t| < p$. Then

$$|x - h|_{\max} \le |x|_{\max};$$

and so, by (2.3), (2.4), and (2.1),

$$|y - wh|_{\max} \le ae^{-cp}|x|_{\max}.$$

This shows that as $p \to \infty$, $wh \to y$. Since wh belongs to the closed linear span of the functions $t^n w$, so does y. The functions y of compact support are dense in C_0, and the proof is complete. □

COMMENT. Let w be a nonnegative function defined on $\mathbb{R}$. The polynomials are said to be complete with respect to the weight w if for each $f \in C(\mathbb{R})$ such that

$$\lim_{|x|\to\infty} w(x)|f(x)| = 0, \tag{2.5}$$

there exists, for each $\varepsilon > 0$, a polynomial P such that

$$w(x)|f(x) - P(x)| < \varepsilon \qquad \text{for all} \quad x \in \mathbb{R}.$$

The problem of finding necessary and sufficient conditions for the polynomials to be complete with respect to w was posed by S.N. Bernstein in 1924 and solved in full generality some thirty years later by S.N. Mergelyan. Mergelyan's beautiful survey article **[M]** contains a complete account of these developments, illustrated with many illuminating examples.

To connect this with the problem considered above, observe that if the polynomials are complete with respect to the positive weight w, then every function

$g \in C_0$ can be approximated uniformly by weighted polynomials. Indeed, $f = g/w$ then satisfies (2.5), and so for each $\varepsilon > 0$, there exists a polynomial P such that

$$|g(x) - w(x)P(x)| = w(x)|f(x) - P(x)| < \varepsilon \quad \text{for all} \quad x \in \mathbb{R}.$$

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2.2. The Müntz Approximation Theorem

According to the Weierstrass approximation theorem, any continuous function $x(t)$ on the interval $[0,1]$ can be approximated uniformly by polynomials in t. Let n be a positive integer. Clearly, if $x(t)$ is continuous on $[0,1]$, so is

$$y(s) = x(s^{1/n}).$$

Now $y(s)$ can be approximated arbitrarily closely in the maximum norm by polynomials $p(s)$. Setting $s = t^n$, we conclude that $x(t)$ can be approximated arbitrarily closely by linear combinations of t^{jn}, $j = 0, 1, \dots$. Thus, not all powers of t are needed in the Weierstrass approximation theorem.

Serge Bernstein posed the problem of determining those sequences of positive numbers $\{\lambda_j\}$ tending to ∞ which have the property that the closed linear span of the functions

$$\{1, t^{\lambda_1}, t^{\lambda_2}, \dots\} \tag{2.6}$$

is the space $C[0,1]$ of all continuous functions on $[0,1]$. After some preliminary results were obtained by Bernstein, Müntz **[M]** proved the following theorem.

THEOREM 2.2. *Let $\{\lambda_j\}$ be a sequence of distinct positive numbers tending to ∞. The functions* (2.6) *span the space $C = C[0,1]$ if and only if*

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j} = \infty. \tag{2.7}$$

PROOF. First we show that if condition (2.7) holds, the functions in (2.6) span C. Let ℓ be a bounded linear functional on C that vanishes on all the functions (2.6):

$$\ell(t^{\lambda_j}) = 0, \qquad j = 1, 2, \dots . \tag{2.8}$$

Let z be a complex variable, $\operatorname{Re} z > 0$. For such z, the function t^z belongs to C and depends analytically on z, in the sense that

$$\lim_{\delta \to 0} \frac{t^{z+\delta} - t^z}{\delta} = (\log t) t^z$$

exists in the norm topology of C. Define

$$f(z) = \ell(t^z). \tag{2.9}$$

Then f is an analytic function of z. Furthermore, since ℓ is bounded (say $\|\ell\| \le 1$) and $|t^z| \le 1$ when $0 \le t \le 1$ and $\operatorname{Re} z > 0$, it follows from (2.9) that

$$|f(z)| \le 1 \qquad \text{for} \qquad \operatorname{Re} z > 0. \tag{2.10}$$