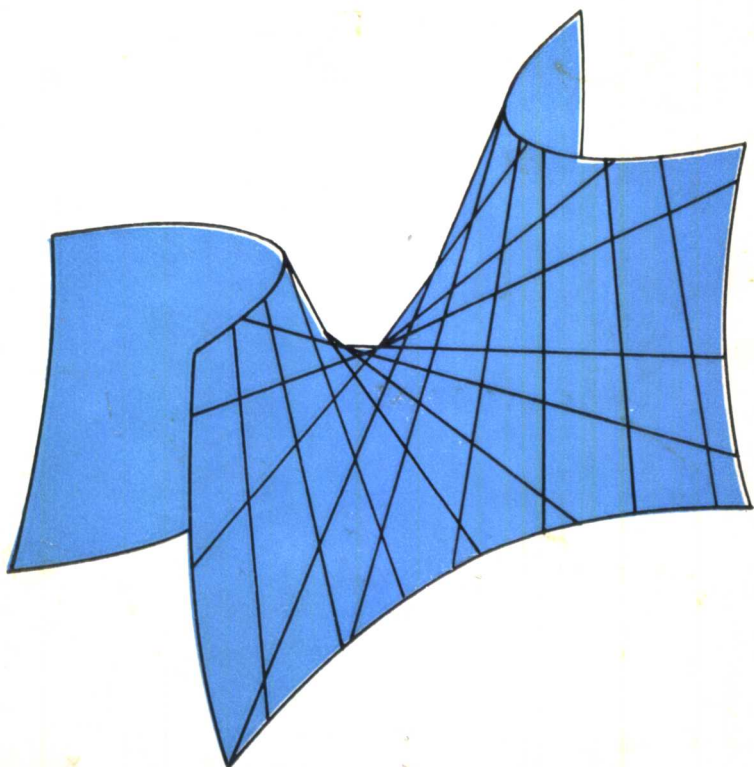


ANALYTIC GEOMETRY

V.A. ILYIN
and
E.G. POZNYAK

MIR
PUBLISHERS
MOSCOW



V. A. Ilyin and E. G. Poznyak

ANALYTIC GEOMETRY

Translated from the Russian
by Irene Aleksanova

Mir Publishers Moscow

First published 1984

Revised from the 1981 Russian edition

На английском языке

© Издательство «Наука», Главная редакция физико-математической литературы, 1981

© English translation, Mir Publishers, 1984

ANALYTIC GEOMETRY



В. А. Ильин, Э. Г. Позняк

АНАЛИТИЧЕСКАЯ ГЕОМЕТРИЯ

МОСКВА «НАУКА»

CONTENTS

Preface	11
Introduction	13

Chapter 1. Systems of Coordinates. The Simplest Problems of Analytic Geometry

1.1. Cartesian Coordinates on a Line	14
1.1.1. Directed segments on an axis	14
1.1.2. Linear operations on directed segments. The basic identity	15
1.1.3. Cartesian coordinates on a straight line	16
1.2. Cartesian Coordinates on a Plane and in Space	17
1.2.1. Cartesian coordinates on a plane	17
1.2.2. Cartesian coordinates in space	18
1.3. The Simplest Problems of Analytic Geometry	19
1.3.1. The concept of a directed segment in space. The projection of a directed segment onto an axis	19
1.3.2. The distance between two points	20
1.3.3. Division of a segment in a given ratio	20
1.3.4. Barycentric coordinates	23
1.4. Polar, Cylindrical, and Spherical Coordinates	24
1.4.1. Polar coordinates	24
1.4.2. Cylindrical coordinates	25
1.4.3. Spherical coordinates	26

Supplement to Chapter 1. Second- and Third-Order Determinants

S1.1. The concepts of a matrix and of a second-order determinant	27
S1.2. A system of two linear equations in two unknowns	28
S1.3. Third-order determinants	30
S1.4. Properties of determinants	31
S1.5. Algebraic adjuncts and minors	33
S1.6. A system of three linear equations in three unknowns with a nonzero determinant	36
S1.7. A homogeneous system of two linear equations in three unknowns	38
S1.8. A homogeneous system of three linear equations in three unknowns	40
S1.9. A nonhomogeneous system of three linear equations in three unknowns with a determinant equal to zero	41

Chapter 2. Vector Algebra

2.1. The Concept of a Vector and Linear Operations on Vectors	43
2.1.1. The concept of a vector	43
2.1.2. Linear operations on vectors	45
2.1.3. The notion of a linear dependence of vectors	50
2.1.4. Linear combinations of two vectors	51
2.1.5. Linear combinations of three vectors	52
2.1.6. Linear dependence of four vectors	54
2.1.7. The concept of a basis. Affine coordinates	55
2.1.8. Projection of a vector onto an axis and its properties	57
2.1.9. The rectangular Cartesian system of coordinates as a special case of the affine system of coordinates	59
2.2. A Scalar Product of Two Vectors	62
2.2.1. Definition of a scalar product	62
2.2.2. Geometrical properties of a scalar product	63
2.2.3. Algebraic properties of a scalar product	63
2.2.4. Expressing a scalar product in Cartesian coordinates	65
2.3. Vector and Mixed Products of Vectors	66
2.3.1. Right-handed and left-handed triads of vectors and systems of coordinates	66
2.3.2. Definition of a vector product of two vectors	67
2.3.3. Geometrical properties of a vector product	67
2.3.4. A mixed product of three vectors	69
2.3.5. Algebraic properties of a vector product	71
2.3.6. Expressing a vector product in Cartesian coordinates	74
2.3.7. Expressing a mixed product in Cartesian coordinates	75
2.3.8. Double vector product	76

Chapter 3. Transformation of Rectangular Cartesian Coordinates on a Plane and in Space. Linear Transformations

3.1. Transformation of Rectangular Cartesian Coordinates on a Plane	79
3.2. Transformation of Rectangular Cartesian Coordinates in Space	83
3.2.1. General formulas for transformation	83
3.2.2. Elucidation of geometrical meaning. Euler's angles	84
3.3. Linear Transformations	88
3.3.1. The concept of a linear transformation of a plane	88
3.3.2. The affine transformations of a plane	88
3.3.3. The basic property of affine transformations of a plane	90
3.3.4. The main invariant of the affine transformation of a plane	92
3.3.5. Affine transformations of space	93
3.3.6. Orthogonal transformations	94
3.4. Projective Transformations	96

Chapter 4. The Equation of a Curve on a Plane. The Equations of a Surface and a Curve in Space

4.1. The Equation of a Curve on a Plane	98
4.1.1. The notion of the equation of a curve	98
4.1.2. Parametric representation of a curve	99

4.1.3.	The equation of a curve in different coordinate systems	101
4.1.4.	Two kinds of problems connected with the analytic representation of a curve	103
4.1.5.	Classification of plane curves	103
4.1.6.	Intersection of two curves	105
4.2.	The Equation of a Surface and the Equations of Curves in Space	106
4.2.1.	The notion of the equation of a surface	106
4.2.2.	Equations of a curve in space	107
4.2.3.	Cylindrical and conic surfaces	108
4.2.4.	Parametric equations of a curve and a surface in space	110
4.2.5.	Classification of surfaces	112
4.2.6.	Intersection of surfaces and curves in space	112
4.2.7.	Concluding remarks	113

Chapter 5. Linear Objects

5.1.	Various Kinds of Equation of a Line on a Plane	114
5.1.1.	General equation of a line	114
5.1.2.	Incomplete equations of a line. An intercept equation of a line	116
5.1.3.	A canonical equation of a line	117
5.1.4.	Parametric equations of a line	118
5.1.5.	A straight line with a slope	118
5.1.6.	An angle between two lines. The conditions of parallelism and perpendicularity of two lines	120
5.1.7.	A normalized equation of a line. Deviation of a point from a line	122
5.1.8.	The equation of a pencil of lines	124
5.2.	Some Problems Concerned with a Line on a Plane	126
5.2.1.	Finding the line passing through the given point $M_1(x_1, y_1)$ and making a given angle φ with the given line $y = k_1x + b_1$	127
5.2.2.	Finding the bisectors of the angles formed by the given lines	128
5.2.3.	The conditions under which the given line meets the given segment AB	128
5.2.4.	Finding the positions of the given point M and of the origin O relative to the angles formed by two given lines	128
5.2.5.	The condition of intersection of three lines at one point	128
5.2.6.	Finding the line passing through the point of intersection of two given lines and satisfying one more condition	130
5.3.	Various Kinds of Equations of a Plane	131
5.3.1.	General equation of a plane	131
5.3.2.	Incomplete equations of a plane. Intercept equation of a plane	132
5.3.3.	An angle between two planes. The conditions of parallelism and perpendicularity of planes	134
5.3.4.	The equation of a plane passing through three distinct points not lying on the same straight line	134
5.3.5.	A normalized equation of a plane. Deviation of a point from a plane	135
5.3.6.	Pencils and bundles of planes	137
5.4.	A Straight Line in Space	138
5.4.1.	Canonical equations of a line in space	138
5.4.2.	Equations of a line passing through two distinct points $M_1(x_1, y_1, z_1)$ and $M_2(x_2, y_2, z_2)$	140
5.4.3.	Parametric equations of a line in space	140
5.4.4.	An angle between lines in space. The conditions of parallelism and perpendicularity of lines	141

5.4.5.	The condition under which two lines belong to the same plane	142
5.4.6.	An angle between a line and a plane. The conditions of parallelism and perpendicularity of a line and a plane	142
5.4.7.	The conditions under which the line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ belongs to the plane $Ax + By + Cz + D = 0$	143
5.4.8.	A bundle of lines	143
5.5.	Some Problems on a Line and a Plane in Space	144
5.5.1.	The condition of intersection of three planes at one and only one point	144
5.5.2.	Finding the bisecting planes of a dihedral angle formed by two given planes	144
5.5.3.	The conditions under which a given plane intersects a given segment AB	144
5.5.4.	Determining the positions of two given points A and B relative to the dihedral angles formed by the given planes	145
5.5.5.	Equations of a line which passes through the given point $M_1(x_1, y_1, z_1)$ and is perpendicular to the given plane $Ax + By + Cz + D = 0$	145
5.5.6.	The equation of a plane which passes through the given point $M_0(x_0, y_0, z_0)$ and is parallel to the given plane $A_1x + B_1y + C_1z + D_1 = 0$	145
5.5.7.	The equation of a plane which passes through the given point $M_0(x_0, y_0, z_0)$ and is perpendicular to the given line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$	145
5.5.8.	The equation of a plane passing through the given line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ and through the given point $M_0(x_0, y_0, z_0)$ not lying on that line	146
5.5.9.	The equation of a plane which passes through the given line $\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ and is parallel to another given line $\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$	146
5.5.10.	The equation of a plane passing through the given line L_1 at right angles to the given plane π	147
5.5.11.	The equation of a perpendicular dropped from the given point M_0 to the given line L_1	147
5.5.12.	Finding the distance from the given point M_0 to the given line L_1	147
5.5.13.	Finding the common perpendicular to two skew lines L_1 and L_2	147
5.5.14.	Finding the shortest distance between two given skew lines L_1 and L_2	147

Chapter 6. Curves of the Second Order

6.1.	Canonical Equations of an Ellipse, a Hyperbola, and a Parabola	148
6.1.1.	An ellipse	149
6.1.2.	A hyperbola	151
6.1.3.	A parabola	152

6.2. Investigating the Shape of an Ellipse, a Hyperbola, and a Parabola Using Their Canonical Equations	154
6.2.1. Investigating the shape of an ellipse	154
6.2.2. Investigating the shape of a hyperbola	155
6.2.3. Investigating the shape of a parabola	159
6.3. Directrices of an Ellipse, a Hyperbola, and a Parabola	160
6.3.1. An eccentricity of an ellipse and of a hyperbola	160
6.3.2. Directrices of an ellipse and a hyperbola	161
6.3.3. The definition of an ellipse and a hyperbola based on their property relative to directrices	166
6.3.4. An ellipse, a hyperbola, and a parabola as conic sections	168
6.3.5. Polar equations of an ellipse, a hyperbola, and a parabola	170
6.4. Tangents to an Ellipse, a Hyperbola, and a Parabola	172
6.4.1. Equations of tangents to an ellipse, a hyperbola, and a parabola	172
6.4.2. Optical properties of an ellipse, a hyperbola, and a parabola	174
6.5. Curves of the Second Order	176
6.5.1. Transformation of the coefficients in the equation for a second-order curve when passing to a new Cartesian system of coordinates	176
6.5.2. Invariants of the equation for a second-order curve. The notion of a type of a second-order curve	179
6.5.3. The centre of a second-order curve	181
6.5.4. A standard simplification of any equation of a second-order curve by a rotation of the axes	183
6.5.5. Simplifying the equation for a central second-order curve of ($I_2 \neq 0$). Classification of central curves	183
6.5.6. Simplifying the equation of a curve of parabolic type ($I_2 = 0$). Classification of curves of parabolic type	187
6.5.7. Reducible second-order curves	189

Chapter 7. Surfaces of the Second Order

7.1. The Notion of a Second-Order Surface	191
7.1.1. Transformation of coefficients in the equation of a second-order surface in transition to a new Cartesian system of coordinates	191
7.1.2. Invariants of the equation of a second-order surface	193
7.1.3. The centre of a second-order surface	194
7.1.4. A standard simplification of any equation of a second-order surface by a rotation of the axes	194
7.2. Classification of Second-Order Surfaces	196
7.2.1. Classification of central surfaces	196
7.2.2. Classification of noncentral surfaces of the second order	199
7.3. Investigating the Shape of Second-Order Surfaces by Using Canonical Equations	201
7.3.1. An ellipsoid	201
7.3.2. Hyperboloids	203
7.3.3. Paraboloids	205
7.3.4. Cones and cylinders of the second order	207
7.3.5. Rectilinear generatrices of second-order surfaces	208

**Appendix. Problems of the Foundations of Geometry and
Substantiation of the Method of Coordinates**

A1. Axioms of Elementary Geometry	212
A1.1. Axioms of incidence	212
A1.2. Axioms of order	213
A1.3. Axioms of congruence	216
A1.4. Axioms of continuity	217
A1.5. Substantiation of the method of coordinates	218
A1.6. The axiom of parallelism	223
A2. The Scheme Used to Prove the Consistency of Euclid's Geometry	224
A3. The Scheme Used to Prove the Consistency of Lobachevsky's Geometry	227
A4. Concluding Remarks on Problems Concerning Systems of Axioms	229
Index	230

PREFACE

The book is based on the lectures delivered by the authors at the Physics Department of Moscow University for a number of years.

Some remarks are due concerning the peculiarity of the exposition. For the first thing, the questions related to the plane and to space are considered in parallel throughout the book.

Vector algebra is treated in considerable detail. The concept of linear relationship of vectors is first introduced and then used to establish the possibility of unique expression of a vector in terms of the affine basis. For proof of the distributive property of vector product and the formulas for double vector product differ from the conventional ones.

To meet the needs of theoretical mechanics, much space is given to the consideration of transformation of rectangular Cartesian coordinates. Elucidating the role of Euler's angles, the authors establish the fact that whatever the two bases of the same orientation, one of them can be transformed into the other by means of parallel displacement and one rotation about some axis in space.

When describing linear objects, besides traditional theoretical material the authors present a large number of problems demonstrating the basic ideas. The discussion of these problems will be of help to students starting on the exercises.

Some space is also given to the questions of the theory of geometrical objects of second order which are of applied nature (optical properties, polar equations and the like).

The Appendix contains material which is not usually presented in traditional courses of analytic geometry. It gives some notion of Hilbert's system of axioms. It also includes the justification of the method of coordinates and some information on the system of

development of principal geometric concepts, on the Euclidean and non-Euclidean geometries and the proof of their consistency. This material is rather urgent both from the point of view of logical principles underlying the construction of geometry and for the understanding of certain divisions of modern physics.

When writing the book we have made wide use of the friendly advice of A. N. Tikhonov and A. G. Sveshnikov to whom we express our deep gratitude.

We are also grateful to N. V. Efimov and A. F. Leont'yev who read the manuscript and made some useful comments.

V. Ilyin and E. Poznyak

INTRODUCTION

Analytic geometry studies the properties of geometric objects with the aid of the analytic method, based on the so-called **method of coordinates**, which was first systematically applied by Descartes*.

The principal concepts of geometry (points, straight lines and planes) belong to the so-called **basic** concepts. They can be described, but every attempt to define each of them inevitably reduces to the replacement of the concept being defined by another one, equivalent to it. From a scientific point of view, a logically faultless method of introduction of the indicated concepts is the **method of axioms**, which was developed and completed by Hilbert**.

The method of axioms is presented in the Appendix at the end of the book. The whole system of axioms of geometry is considered there as well as the so-called **non-Euclidean geometry**, which results from replacement of one of the axioms (the so-called **parallel axiom**) by the assertion negating it.

The question of **consistency** of both the Euclidean and the non-Euclidean geometry is considered there and it is established that a specific realization of the collection of objects satisfying the axioms of geometry is the introduction of points as various ordered triples (x, y, z) of real numbers, of straight lines as sets of triples (x, y, z) satisfying a system of two linear equations, and of planes as sets of triples (x, y, z) satisfying one linear equation.

The method of axioms lays the foundation for the method of coordinates on which analytic geometry is based. Thus, for instance, the question concerning the possibility of introducing coordinates on a straight line follows from the possibility of *establishing a one-to-one correspondence between the set of all points of a line and the set of all real numbers*. The proof of this possibility is based on the axioms of geometry and on the axioms (properties) of a set of real numbers*** and is given in the Appendix.

Thus, the Appendix contains the justification of the system of development of the principal geometric notions and of the method of coordinates underlying analytic geometry.

The method of coordinates is a powerful apparatus making it possible to apply methods of algebra and mathematical analysis to investigation of geometric objects.

* René Descartes (1596-1650), a great French mathematician and philosopher.

** David Hilbert (1862-1943), a great German mathematician.

*** The properties of real numbers and the axioms method for introducing a set of real numbers are given in Chapter 2 and in the Appendix to our book *Fundamentals of Mathematical Analysis*, Part 1, Mir Publishers, Moscow (1982).

Chapter 1

SYSTEMS OF COORDINATES.

THE SIMPLEST PROBLEMS OF ANALYTIC GEOMETRY

This chapter deals with the Cartesian coordinates* on a straight line, on a plane, and in space. It also includes the simplest problems of analytic geometry (the distance between two points, division of a segment in a given ratio) and gives some notion of other systems of coordinates (polar, cylindrical, and spherical).

1.1. Cartesian Coordinates on a Line

1.1.1. Directed segments on an axis. A straight line** with the direction indicated on it is called an *axis*. A segment on an axis is said to be *directed* if it is indicated which of its boundary points is its beginning and which is its end. We shall designate the directed segment beginning at a point A and terminating at a point B by the symbol \overrightarrow{AB} (Fig. 1.1 shows the directed segments \overrightarrow{AB} and \overrightarrow{CD}). We shall also consider the so-called *zero directed segments* whose initial and terminal points coincide.

Every directed segment is associated with its numerical characteristic, the so-called *magnitude of the directed segment*. The *magnitude* AB of the directed segment \overrightarrow{AB} is a number equal to the length of the segment \overrightarrow{AB} taken with the plus sign if its direction coincides with that of the axis, and with the minus sign if its direction is

* *Coordinates* (from the Latin words *co* meaning jointly and *ordinatus* meaning ordered, definite) are numbers whose specification defines the position of a point on a straight line, on a plane or in space (on a line or on a surface, respectively). The *method of coordinates* was introduced by the French scientist René Descartes. This method makes it possible to interpret geometrical problems in the language of mathematical analysis and, conversely, to give geometrical interpretation to facts of analysis.

** The Appendix at the end of the book includes axiomatic introduction of the principal geometric notions (points, lines, planes). A relationship is also established there between the geometric notion of a *straight line* and the notion of a *number axis* (see Ilyin, Poznyak, *Fundamentals of Mathematical Analysis*, Part 1, Mir Publishers, Moscow).

opposite to that of the axis. The magnitudes of all zero directed segments are taken to be zero.

1.1.2. Linear operations on directed segments. The basic identity. We first define the equality of directed segments. We shall displace the directed segments along the axis on which they lie retaining their length and direction*.

Two nonzero directed segments are said to be equal if their terminal points coincide when their initial points are brought into coincidence.

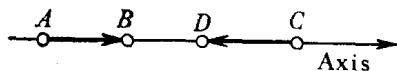


Fig. 1.1

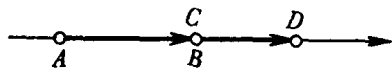


Fig. 1.2

Any two zero directed segments are considered to be equal.

It is evident that the necessary and sufficient condition for equality of two directed segments on a given axis is the equality of the magnitudes of those segments.

The term *linear operations* on directed segments will be used for operations of addition of such segments and of multiplication of a directed segment by a real number.

Let us now define these operations.

To define the **sum** of the directed segments \overrightarrow{AB} and \overrightarrow{CD} , we bring the initial point C of the segment \overrightarrow{CD} into coincidence with the terminal point B of the segment \overrightarrow{AB} (Fig. 1.2). The resulting directed segment \overrightarrow{AD} is called the **sum** of the segments \overrightarrow{AB} and \overrightarrow{CD} and is designated as $\overrightarrow{AB} + \overrightarrow{CD}$.

The following fundamental theorem holds true.

Theorem 1.1. *The magnitude of the sum of directed segments is equal to the sum of the magnitudes of the segments being added.*

Proof. Suppose at least one of the segments \overrightarrow{AB} and \overrightarrow{CD} is zero. If, say, \overrightarrow{CD} is zero, then the sum $\overrightarrow{AB} + \overrightarrow{CD}$ coincides with the segment \overrightarrow{AB} and the statement of the theorem is true. Assume now that both segments \overrightarrow{AB} and \overrightarrow{CD} are nonzero. Let us bring the initial point C of \overrightarrow{CD} into coincidence with the terminal point B of \overrightarrow{AB} . Then we have $\overrightarrow{AB} + \overrightarrow{CD} = \overrightarrow{AD}$. We have to prove the validity of

* The question as to the possibility of displacing segments is connected with the congruency axioms (see the Appendix and, in particular, a footnote on p. 216).

the equality $\overrightarrow{AB} + \overrightarrow{CD} = \overrightarrow{AD}$. Let us consider the case when the segments \overrightarrow{AB} and \overrightarrow{CD} are of the same direction (Fig. 1.2). Then the length of \overrightarrow{AD} is equal to the sum of the lengths of the segments \overrightarrow{AB} and \overrightarrow{CD} and, besides, the direction of \overrightarrow{AD} coincides with the direction of each of the segments \overrightarrow{AB} and \overrightarrow{CD} . Therefore, the equality $\overrightarrow{AB} + \overrightarrow{CD} = \overrightarrow{AD}$ in question is true. Let us now consider one more

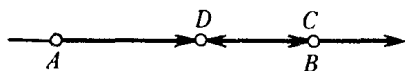


Fig. 1.3

possible case when the segments \overrightarrow{AB} and \overrightarrow{CD} are of opposite directions (Fig. 1.3). In that case, the magnitudes of the segments \overrightarrow{AB} and \overrightarrow{CD}

are of unlike signs and, therefore, the length of the segment \overrightarrow{AD} is $|\overrightarrow{AB} + \overrightarrow{CD}|$. Since the direction of the segment \overrightarrow{AD} coincides with that of the longer of the segments \overrightarrow{AB} and \overrightarrow{CD} , the sign of the magnitude of the segment \overrightarrow{AD} coincides with the sign of the number $\overrightarrow{AB} + \overrightarrow{CD}$, that is, the equality $\overrightarrow{AB} + \overrightarrow{CD} = \overrightarrow{AD}$ holds true. We have proved the theorem.

Corollary. For any arrangement of the points, A, B, C on the number axis the magnitudes of the directed segments \overrightarrow{AB} , \overrightarrow{BC} , and \overrightarrow{AC} satisfy the relation

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}, \quad (1.1)$$

which is called the basic identity.

The operation of multiplication of a directed segment by the real number α is defined as follows.

The product of the directed segment \overrightarrow{AB} by the number α is a directed segment, designated as $\alpha \cdot \overrightarrow{AB}$, whose length is equal to the product of the number $|\alpha|$ by the length of the segment \overrightarrow{AB} and whose direction coincides with that of the segment \overrightarrow{AB} for $\alpha > 0$ and is opposite to it for $\alpha < 0$.

The magnitude of the directed segment $\alpha \cdot \overrightarrow{AB}$ is, evidently, equal to $\alpha \cdot \overrightarrow{AB}$.

1.1.3. Cartesian coordinates on a straight line. The Cartesian coordinates on a line are introduced as follows. We choose a definite direction and some point O (the origin) on a line* (Fig. 1.4). In addi-

* Recall that a straight line with the direction indicated on it is called an axis.