# ANALYTIC GEOMETRY

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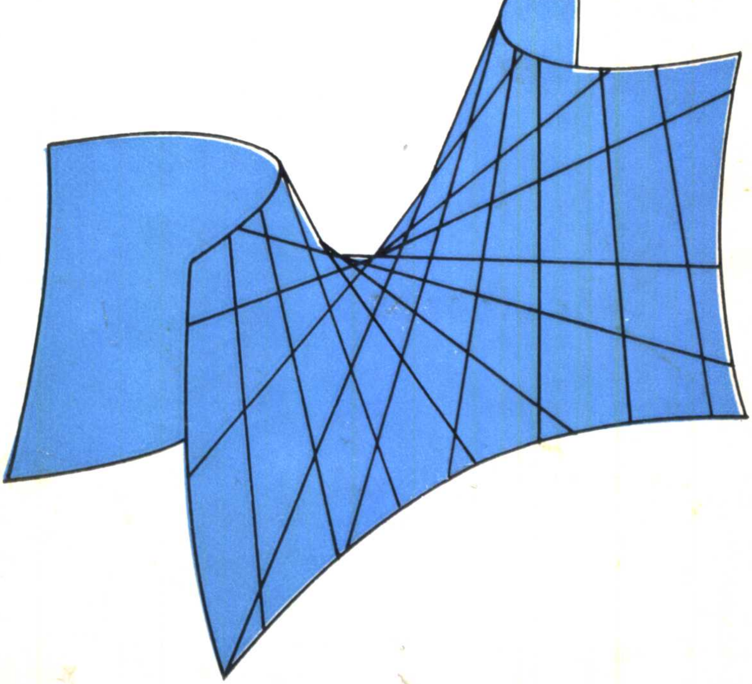

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# ANALYTIC GEOMETRY

**В. А. Ильин, Э. Г. Позняк**

# АНАЛИТИЧЕСКАЯ ГЕОМЕТРИЯ

МОСКВА «НАУКА»

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# PREFACE

The book is based on the lectures delivered by the authors at the Physics Department of Moscow University for a number of years.

Some remarks are due concerning the peculiarity of the exposition. For the first thing, the questions related to the plane and to space are considered in parallel throughout the book.

Vector algebra is treated in considerable detail. The concept of linear relationship of vectors is first introduced and then used to establish the possibility of unique expression of a vector in terms of the affine basis. For proof of the distributive property of vector product and the formulas for double vector product differ from the conventional ones.

To meet the needs of theoretical mechanics, much space is given to the consideration of transformation of rectangular Cartesian coordinates. Elucidating the role of Euler's angles, the authors establish the fact that whatever the two bases of the same orientation, one of them can be transformed into the other by means of parallel displacement and one rotation about some axis in space.

When describing linear objects, besides traditional theoretical material the authors present a large number of problems demonstrating the basic ideas. The discussion of these problems will be of help to students starting on the exercises.

Some space is also given to the questions of the theory of geometrical objects of second order which are of applied nature (optical properties, polar equations and the like).

The Appendix contains material which is not usually presented in traditional courses of analytic geometry. It gives some notion of Hilbert's system of axioms. It also includes the justification of the method of coordinates and some information on the system of development of principal geometric concepts, on the Euclidean and non-Euclidean geometries and the proof of their consistency. This material is rather urgent both from the point of view of logical principles underlying the construction of geometry and for the understanding of certain divisions of modern physics.

When writing the book we have made wide use of the friendly advice of A. N. Tikhonov and A. G. Sveshnikov to whom we express our deep gratitude.

We are also grateful to N. V. Efimov and A. F. Leont'yev who read the manuscript and made some useful comments.

*V. Ilyin and E. Poznyak*

## INTRODUCTION

Analytic geometry studies the properties of geometric objects with the aid of the analytic method, based on the so-called **method of coordinates,** which was first systematically applied by Descartes*.

The principal concepts of geometry (points, straight lines and planes) belong to the so-called **basic** concepts. They can be described, but every attempt to define each of them inevitably reduces to the replacement of the concept being defined by another one, equivalent to it. From a scientific point of view, a logically faultless method of introduction of the indicated concepts is the **method of axioms,** which was developed and completed by Hilbert**.

The method of axioms is presented in the Appendix at the end of the book. The whole system of axioms of geometry is considered there as well as the so-called **non-Euclidean geometry**, which results from replacement of one of the axioms (the so-called **parallel axiom**) by the assertion negating it.

The question of **consistency** of both the Euclidean and the non-Euclidean geometry is considered there and it is established that a specific realization of the collection of objects satisfying the axioms of geometry is the introduction of points as various ordered triples $(x, y, z)$ of real numbers, of straight lines as sets of triples $(x, y, z)$ satisfying a system of two linear equations, and of planes as sets of triples $(x, y, z)$ satisfying one linear equation.

The method of axioms lays the foundation for the method of coordinates on which analytic geometry is based. Thus, for instance, the question concerning the possibility of introducing coordinates on a straight line follows from the possibility *of establishing a one-to-one correspondence between the set of all points of a line and the set of all real numbers.* The proof of this possibility is based on the axioms of geometry and on the axioms (properties) of a set of real numbers*** and is given in the Appendix.

Thus, the Appendix contains the justification of the system of development of the principal geometric notions and of the method of coordinates underlying analytic geometry.

The method of coordinates is a powerful apparatus making it possible to apply methods of algebra and mathematical analysis to investigation of geometric objects.

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* René Descartes (1596-1650), a great French mathematician and philosopher.

** David Hilbert (1862-1943), a great German mathematician.

*** The properties of real numbers and the axioms method for introducing a set of real numbers are given in Chapter 2 and in the Appendix to our book *Fundamentals of Mathematical Analysis*, Part 1, Mir Publishers, Moscow (1982).

## Chapter 1

# SYSTEMS OF COORDINATES. THE SIMPLEST PROBLEMS OF ANALYTIC GEOMETRY

This chapter deals with the Cartesian coordinates* on a straight line, on a plane, and in space. It also includes the simplest problems of analytic geometry (the distance between two points, division of a segment in a given ratio) and gives some notion of other systems of coordinates (polar, cylindrical, and spherical).

## 1.1. Cartesian Coordinates on a Line

**1.1.1. Directed segments on an axis.** A straight line** with the direction indicated on it is called an *axis*. A segment on an axis is said to be *directed* if it is indicated which of its boundary points is its beginning and which is its end. We shall designate the directed segment beginning at a point $A$ and terminating at a point $B$ by the symbol $\overrightarrow{AB}$ (Fig. 1.1 shows the directed segments $\overrightarrow{AB}$ and $\overrightarrow{CD}$). We shall also consider the so-called *zero directed segments* whose initial and terminal points coincide.

Every directed segment is associated with its numerical characteristic, the so-called *magnitude of the directed segment*. The *magnitude* $AB$ of the directed segment $\overrightarrow{AB}$ is a number equal to the length of the segment $\overrightarrow{AB}$ taken with the plus sign if its direction coincides with that of the axis, and with the minus sign if its direction is

---

* *Coordinates* (from the Latin words *co* meaning jointly and *ordinatus* meaning ordered, definite) are numbers whose specification defines the position of a point on a straight line, on a plane or in space (on a line or on a surface, respectively). The *method of coordinates* was introduced by the French scientist René Descartes. This method makes it possible to interpret geometrical problems in the language of mathematical analysis and, conversely, to give geometrical interpretation to facts of analysis.

** The Appendix at the end of the book includes axiomatic introduction of the principal geometric notions (points, lines, planes). A relationship is also established there between the geometric notion of a *straight line* and the notion of a *number axis* (see Ilyin, Poznyak, *Fundamentals of Mathematical Analysis*, Part 1, Mir Publishers, Moscow).

opposite to that of the axis. The magnitudes of all zero directed segments are taken to be zero.

**1.1.2. Linear operations on directed segments. The basic identity.** We first define the equality of directed segments. We shall displace the directed segments along the axis on which they lie retaining their length and direction*.

*Two nonzero directed segments are said to be equal if their terminal points coincide when their initial points are brought into coincidence.*

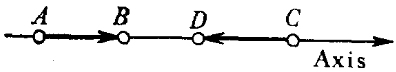


Fig. 1.1

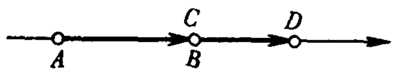


Fig. 1.2

*Any two zero directed segments are considered to be equal.*

It is evident that *the necessary and sufficient condition for equality of two directed segments on a given axis is the equality of the magnitudes of those segments.*

The term *linear operations* on directed segments will be used for *operations of addition* of such segments and of *multiplication of a directed segment by a real number.*

Let us now define these operations.

To define the **sum** of the directed segments $\overrightarrow{AB}$ and $\overrightarrow{CD}$, we bring the initial point $C$ of the segment $\overrightarrow{CD}$ into coincidence with the terminal point $B$ of the segment $\overrightarrow{AB}$ (Fig. 1.2). The resulting directed segment $\overrightarrow{AD}$ is called the *sum* of the segments $\overrightarrow{AB}$ and $\overrightarrow{CD}$ and is designated as $\overrightarrow{AB} + \overrightarrow{CD}$.

The following fundamental theorem holds true.

**Theorem 1.1.** *The magnitude of the sum of directed segments is equal to the sum of the magnitudes of the segments being added.*

*Proof.* Suppose at least one of the segments $\overrightarrow{AB}$ and $\overrightarrow{CD}$ is zero. If, say, $\overrightarrow{CD}$ is zero, then the sum $\overrightarrow{AB} + \overrightarrow{CD}$ coincides with the segment $\overrightarrow{AB}$ and the statement of the theorem is true. Assume now that both segments $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are nonzero. Let us bring the initial point $C$ of $\overrightarrow{CD}$ into coincidence with the terminal point $B$ of $\overrightarrow{AB}$. Then we have $\overrightarrow{AB} + \overrightarrow{CD} = \overrightarrow{AD}$. We have to prove the validity of

* The question as to the possibility of displacing segments is connected with the congruency axioms (see the Appendix and, in particular, a footnote on p. 216).

the equality $AB + CD = AD$. Let us consider the case when the segments $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are of the same direction (Fig. 1.2). Then the length of $\overrightarrow{AD}$ is equal to the sum of the lengths of the segments $\overrightarrow{AB}$ and $\overrightarrow{CD}$ and, besides, the direction of $\overrightarrow{AD}$ coincides with the direction of each of the segments $\overrightarrow{AB}$ and $\overrightarrow{CD}$. Therefore, the equality $AB + CD = AD$ in question is true. Let us now consider one more possible case when the segments $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are of opposite directions (Fig. 1.3). In that case, the magnitudes of the segments $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are of unlike signs and, therefore, the length of the segment $\overrightarrow{AD}$ is $|AB + CD|$. Since the direction of the segment $\overrightarrow{AD}$ coincides with that of the longer of the segments $\overrightarrow{AB}$ and $\overrightarrow{CD}$, the sign of the magnitude of the segment $\overrightarrow{AD}$ coincides with the sign of the number $AB + CD$, that is, the equality $AB + CD = AD$ holds true. We have proved the theorem.

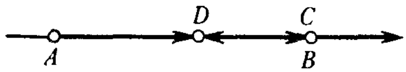

Fig. 1.3

**Corollary.** *For any arrangement of the points, A, B, C on the number axis the magnitudes of the directed segments* $\overrightarrow{AB}$, $\overrightarrow{BC}$, *and* $\overrightarrow{AC}$ *satisfy the relation*

$$AB + BC = AC, \tag{1.1}$$

*which is called the basic identity.*

The operation of multiplication of a directed segment by the real number $\alpha$ is defined as follows.

*The product of the directed segment* $\overrightarrow{AB}$ *by the number* $\alpha$ *is a directed segment, designated as* $\alpha \cdot \overrightarrow{AB}$, *whose length is equal to the product of the number* $|\alpha|$ *by the length of the segment* $\overrightarrow{AB}$ *and whose direction coincides with that of the segment* $\overrightarrow{AB}$ *for* $\alpha > 0$ *and is opposite to it for* $\alpha < 0$.

The magnitude of the directed segment $\alpha \cdot \overrightarrow{AB}$ is, evidently, equal to $\alpha \cdot AB$.

**1.1.3. Cartesian coordinates on a straight line.** The Cartesian coordinates on a line are introduced as follows. We choose a definite direction and some point $O$ (the origin) on a line* (Fig. 1.4). In addi-

* Recall that a straight line with the direction indicated on it is called an axis.