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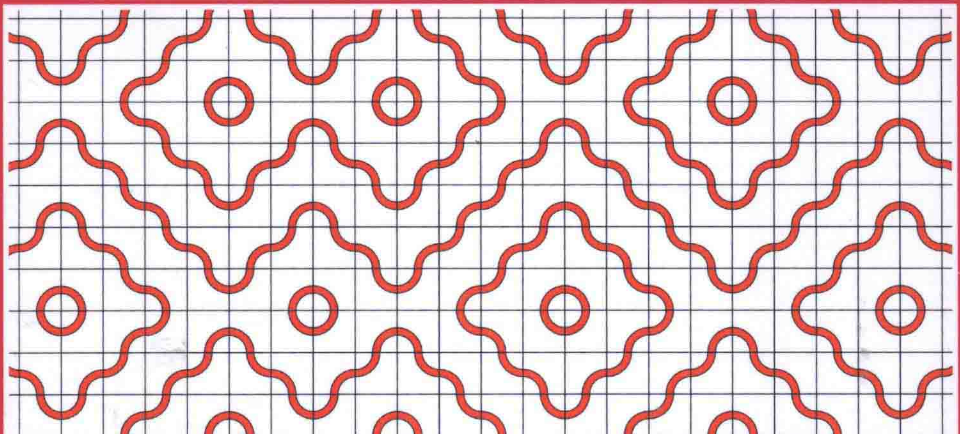
COMBINATORICS, WORDS AND SYMBOLIC DYNAMICS

Edited by Valérie Berthé and Michel Rigo



ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

Combinatorics, Words and Symbolic Dynamics

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COMBINATORICS, WORDS AND SYMBOLIC DYNAMICS

Internationally recognised researchers look at developing trends in combinatorics with applications in the study of words and in symbolic dynamics. They explain the important concepts, providing a clear exposition of some recent results, and emphasise the emerging connections between these different fields. Topics include combinatorics on words, pattern avoidance, graph theory, tilings and theory of computation, multidimensional subshifts, discrete dynamical systems, ergodic theory, numeration systems, dynamical arithmetics, automata theory and synchronised words, analytic combinatorics, continued fractions and probabilistic models. Each topic is presented in a way that links it to the main themes, but then they are also extended to repetitions in words, similarity relations, cellular automata, friezes and Dynkin diagrams.

The book will appeal to graduate students, research mathematicians and computer scientists working in combinatorics, theory of computation, number theory, symbolic dynamics, tilings and stringology. It will also interest biologists using text algorithms.

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Preface

Inspired by the celebrated Lothaire series (Lothaire, 1983, 2002, 2005) and animated by the same spirit as in the book (Berthé and Rigo, 2010), this collaborative volume aims at presenting and developing recent trends in *combinatorics* with applications in the study of *words* and in *symbolic dynamics*.

On the one hand, some of the newest results in these areas have been selected for this volume and here benefit from a synthetic exposition. On the other hand, emphasis on the connections existing between the main topics of the book is sought. These connections arise, for instance, from numeration systems that can be associated with algorithms or dynamical systems and their corresponding expansions, from cellular automata and the computation or the realisation of a given entropy, or even from the study of friezes or from the analysis of algorithms.

This book is primarily intended for graduate students or research mathematicians and computer scientists interested in combinatorics on words, pattern avoidance, graph theory, quivers and frieze patterns, automata theory and synchronised words, tilings and theory of computation, multidimensional subshifts, discrete dynamical systems, ergodic theory and transfer operators, numeration systems, dynamical arithmetics, analytic combinatorics, continued fractions, probabilistic models. We hope that some of the chapters can serve as useful material for lecturing at master/graduate level. Some chapters of the book can also be interesting to biologists and researchers interested in text algorithms or bio-informatics.

Let us succinctly sketch the general landscape of the volume. Short abstracts of each chapter can be found below. The book can roughly be divided into four general blocks. The first one, made of Chapters 2 and 3, is devoted to *numeration systems*. The second block, made of Chapters 4 to 6, pertains to *combinatorics of words*. The third block is concerned with *symbolic dynamics*: in the one-dimensional setting with Chapter 7, and in the multidimensional one, with Chapters 8 and 9. The last block, made of Chapters 10 and 11, has again a *combinatorial nature*.

Words, i.e., finite or infinite sequences of symbols taking values in a finite set, are ubiquitous in the sciences. It is because of their strong representation power: they arise as a natural way to code elements of an infinite set using finitely many symbols. So let us start our general description with **combinatorics on words**. *Powers, repetitions and periods* have been at the core of this field since its birth with the work of Thue (1906a, 1912). Thue's work has been fruitfully extended to several important research directions. Let us mention the notion of abelian repetition (introduced by Paul Erdős), and the notion of fractional repetition (introduced by Françoise Dejean) leading to a famous conjecture on repetition threshold that was recently proved in 2009. We have chosen to focus on these fundamental notions in Chapters 4, 5 and 6 devoted to words. Both application-driven and theoretical viewpoints are presented. Note that the systematic study of repetitions covers a wide field of applications ranging from number theory to bio-informatics. As striking examples, let us quote the work of Novikov and Adian (1968) on the Burnside problem for groups and the work of Adamczewski and Bugeaud (2007) on the transcendence of real numbers.

Chapter 4 is focused on avoidable regularities in words, which consists of avoiding some types of repetitions. This chapter also covers the use of non-effective *probabilistic methods* like the Lovász local lemma and introduces some decision problems about automatic sequences. Interestingly, Büchi's theorem from 1960 and first-order logic are important tools leading to decision procedures for instances of combinatorial problems that can be expressed in an extension of the Presburger arithmetic. As an example, one can get an automated certification that the Thue–Morse word is overlapfree.

Chapter 5 deals with *redundancies* in textual data and is also built around the analysis of periodicity in words but aimed towards applications, in particular considering text algorithms, text compression, algorithms for bio-informatics, and analysis of biological sequences. It presents several methods used to detect periodicity, like the one used for compression in the Lempel–Ziv factorisation.

Similarity relations on words are considered in Chapter 6 from the two perspectives of periodicity and repetition freeness. A similarity relation on words is induced by a compatibility relation on letters assumed to be a reflexive and symmetric relation. Two words are similar if they are of the same length and their corresponding letters are pairwise compatible. Similarity relations generalise the notion of partial words, i.e., words where a do-not-know symbol $\diamond$ may be used. As an example, the word $a \diamond ba\diamond$ is compatible with the word *abbab*. These relations can be seen as a model for inaccurate information on words. It is motivated here again both by theoretical issues and by applications arising from computer science (e.g., string matching) and molecular biology.

Combinatorial problems do not only occur in this book in the framework of words but also in a more general setting of **algebraic** and **analytic combinatorics**, in particular with Chapters 10 and 11 which are combinatorial in nature (of course, general combinatorial tools such as formal power series occur in many chapters). Chapter 10 belongs to *algebraic combinatorics* through the study of a class of sequences of natural numbers associated with certain quivers (directed graphs). Quivers are commonly used in representation theory. It is possible to associate a numerical frieze with a quiver. These are integer-valued sequences. They are an extension to the whole plane of the *frieze patterns* introduced by Coxeter (1971). These recursions, although highly non-linear, produce sometimes rational, or even $\mathbb{N}$-rational, sequences. The question of rationality of the friezes is the central question which will be answered in this chapter: if friezes are rational, they must be $\mathbb{N}$-rational. This chapter also introduces the notion of SL_2-tilings and their applications. An SL_2-tiling of the plane is a filling of $\mathbb{Z}^2$ by numbers in such a way that each adjacent two by two minor is equal to 1.

Chapter 11 belongs to the framework of *dynamical analysis of algorithms*. It focuses on the study of random Kronecker sequences through their discrepancy and their Arnold constant. It thus provides an illustration of the use of probabilistic methods in combinatorics by applying to Kronecker sequences the dynamical analysis methodology, which is a mixing between analysis of algorithms and dynamical systems relying on spectral properties of transfer operators. Recall that the use of combinatorics in the analysis of algorithms, initiated by D. E. Knuth, greatly relies on number theory, asymptotic methods and computer use. Let us also mention recent successful applications of the dynamical approach in the analysis of algorithms in connection with number theory through the analysis of the Gauss map as illustrated, e.g., in the work of Baladi and Vallée (2005).

Combinatorics on words and **symbolic dynamics** are intimately related. Indeed, the coding of orbits and trajectories by words over a (finite) alphabet constitutes the basis of symbolic dynamical systems. Recall that a discrete dynamical system is a continuous map defined on a compact metric space X onto itself. It is therefore natural to code trajectories of points in the state space using a (finite) partition of X. One thus gets infinite words as codings and the corresponding dynamical systems are said to be symbolic. The study of symbolic dynamical systems in a multidimensional setting has recently given rise to striking results intertwining computational complexity, entropy, ergodic theory and topological dynamics, such as, e.g., in the work of Hochman and Meyerovitch (2010). Theory of recursion appears to be a key tool in the study of finite type or sofic multidimensional shifts: it appears clearly that many properties of dynamical systems can thus be described in terms of recursion theory.

Classical symbolic dynamics is linked with graph theory and automata theory. This is the heart of Chapter 7 with the study of the celebrated "*Road Colouring Problem*". This latter problem is a classical question about synchronisation in an automaton (or a graph). A synchronising word maps every state of an automaton to the same state. An automaton is synchronised if it has a synchronising word. The *Road Colouring Theorem* states that every complete deterministic automaton with an aperiodic directed underlying graph has the same graph as a synchronised automaton. This result was first conjectured by Adler *et al.* (1977) and has been recently solved by Trahtman (2009). The aims of Chapter 7 are to present a proof and an efficient algorithm for this problem and also to consider links existing to another famous conjecture in automata theory, namely the *Černý Conjecture*. This conjecture asserts that a synchronised deterministic automaton with n states has a synchronising word of length at most $(n-1)^2$.

Chapters 8 and 9 are articulated on *multidimensional symbolic dynamics* by stressing the striking fundamental differences with the one-dimensional case. One of the main features of the multidimensional case is that computations can be implemented in subshifts of finite type. Indeed, it is possible to construct for a given Turing machine T, a shift of finite type in which every configuration represents arbitrarily long computations. This yields undecidability for problems that were clearly decidable in the one-dimensional case. Striking connections with complexity, computability and decidability issues are presented.

Chapter 8 focuses on cellular automata whereas Chapter 9 focuses on shifts of finite type. Both chapters are complementary and intertwined. Chapter 8 is organised around the notion of tiling linked with cellular automata. Again fundamental differences exist between one-dimensional cellular automata and multidimensional ones. These differences may be explained by the theory of tilings, like the existence of aperiodic tilings or the fact that the so-called domino problem is undecidable. Chapter 9 is more precisely aimed at connections between combinatorial dynamical systems and effective systems, and, in particular, at aspects of multidimensional symbolic dynamics and cellular automata, including realisation theorems. Thus will be presented the characterisation of the entropy for multidimensional shifts of finite type in terms of computable real numbers proved by Hochman and Meyerovitch (2010).

Let us conclude our brief presentation of the book with **numeration systems**. In a generic way, a numeration system allows the *expansion* of numbers as words over an alphabet of digits. A numeration system usually is either defined by an algorithm providing expansions, or by an iterative process associated with a dynamical system. So again, words are demonstrating their representation power. Amongst the various questions related to the expansions of numbers, we have chosen to develop two focused viewpoints on numeration systems with non-integer bases.

Chapter 2 deals with the possible expansions of a number that can occur when the base and the alphabet are fixed. It concentrates on the cases where such an expansion is unique. The viewpoint on numeration provided by Chapter 3 is of an arithmetic nature and relies on the notion of *mediety*. A mediety is a binary operation that allows us to split an interval into two smaller ones and to repeat the process. Assuming that any infinite sequence of such successive intervals decreases to a single number, we get a coding of any element of an initial interval by an infinite word over a binary alphabet. Chapter 3 revisits, under the viewpoint of medieties, various classical codings and representations such as the numeration in base 2, or continued fractions, classic ones as well as Rosen ones with k-medieties.

Parts of the material presented in this book were presented during the CANT school that was organised at the Centre International de Rencontres Mathématiques

(CIRM) from 21st to 25th May 2012 in Marseille. We thank the CIRM for supporting this event that gathered more than one hundred participants from eighteen countries.

We now give a short abstract of every chapter of the book. Chapter 1 is a general introduction where the main notions that will occur in this book are presented. The reader may skip this chapter in a first reading and use it as a reference if needed. Let us now move to the main contributions of this book listed by order of appearance.

Chapter 2 by M. de Vries and V. Komornik
Expansions in non-integer bases

The familiar integer base expansions were extended to non-integer bases in a seminal paper of Rényi in 1957. Since then many surprising phenomena were discovered and a great number of papers were devoted to unexpected connections with probability and ergodic theory, combinatorics, symbolic dynamics, measure theory, topology and number theory. For example, although a number cannot have more than two expansions in integer bases, in non-integer bases a number has generically a continuum of expansions. Despite this generic situation, Erdős *et al.* discovered in 1990 some unexpected uniqueness phenomena which gave a new impetus to this research field. The purpose of Chapter 2 is to give an overview of parts of this rich theory. The authors present a number of elementary but powerful proofs and give many examples. Some proofs presented here are new.

Chapter 3 by B. Rittaud
Medieties, end-first algorithms, and the case of Rosen continued fractions

A *mediety* is any rule that splits a given initial interval $\mathbb{I}$ in two subintervals, then also these intervals into two subintervals, etc., such that any decreasing sequence of such subintervals reduces to a single element. Example of medieties are the arithmetic mean, that gives rise to the base 2-numeration system, and the mediant, from which the theory of continued fractions can be recovered. Engel continued fractions provide another example.

Chapter 3 investigates some general properties of medieties, as for example the question of the numbers that are approximated the most slowly by elements of the set $\mathbb{F}$ of bounds of intervals defined by the mediety. For example, it is well-known that the Golden Ratio is the number the most slowly approximated by rational numbers, and this property corresponds to the case of the mediant, for which $\mathbb{F} = \mathbb{Q}^+$. This chapter also introduces some *end-first algorithms*, that is, algorithms that provide the coding of any element of $\mathbb{F}$ by the mediety starting from its end instead of its beginning. In the case of the mediant, these algorithms are related to random Fibonacci sequences.

Last, Chapter 3 presents *k-medieties*, that is, medieties that split intervals into k

subintervals. In particular, it shows that λ_k-Rosen continued fractions, i.e., continued fractions in which the partial quotients are integral multiples of $\lambda_k := 2\cos(\pi/k)$ (for an integer $k \geq 3$), derive from a mediety that generalises the mediant in a similar way that the base k generalises the binary numeration system.

Chapter 4 by N. Rampersad and J. Shallit
Repetitions in words

Avoidable repetitions in words are discussed. Chapter 4 begins by a brief overview of the avoidability of the classical patterns, such as squares, cubes and overlaps. The authors describe the most common technique used to construct infinite words avoiding these kinds of patterns, namely, the use of iterated morphisms. They also describe a probabilistic approach to avoidability based on the Lovász local lemma. Next, they consider generalisations of the classical patterns, such as fractional powers (which leads naturally to a discussion of Dejean's theorem), repetitions in arithmetic progressions and abelian repetitions. Some methods for counting, or at least estimating, the number of words of a given length avoiding a pattern or set of patterns are also presented in Chapter 4. Finally, the authors briefly explain an algorithmic method for obtaining computer-assisted proofs of certain types of results on automatic sequences.

Chapter 5 by G. Badkobeh, M. Crochemore, C. S. Iliopoulos and M. Kubica
Text redundancies

In relation with the previous chapter, Chapter 5 deals with several types of redundancies occurring in textual data. Detecting them in texts is essential in applications like pattern matching, text compression or further to extract patterns for data mining. Considered redundancies include repetitions, word powers, maximal periodicities, repeats, palindromes, and their extension to notions of covers and seeds. Main results like lower and upper bounds as well as detection algorithms on some patterns are reported.

Chapter 6 by V. Halava, T. Harju and T. Kärki
Similarity relations on words

The authors of Chapter 6 consider similarity relations on words that were originally introduced in order to generalise partial word, i.e., words with a do-not-know symbol. A similarity relation on words is induced by a compatibility relation on letters assumed to be a reflexive and symmetric relation. In connection with the previous two chapters, similarity of words is studied from two perspectives that are central in combinatorics on words: periodicity and repetition freeness. In particular variations of Fine and Wilf's theorem are stated to witness interaction properties between the extended notions of periodicity induced by similarity relation. Also, squarefreeness of words is defined for relational words and tight bounds for the repetition thresholds are given.

Chapter 7 by M.-P. Béal and D. Perrin
Synchronised automata

A survey of results concerning synchronised automata is presented in Chapter 7. The authors first discuss the state of art concerning the Černý Conjecture on the minimal length of synchronising words. They next describe the case of circular automata and more generally one-cluster automata. A proof of the Road Colouring Theorem is also presented in Chapter 7.

Chapter 8 by J. Kari
Cellular automata, tilings and (un)computability

Chapter 8 reviews some basic concepts and results on the theory of cellular automata. Algorithmic questions concerning cellular automata and tilings are also discussed. Covered topics include injectivity and surjectivity properties, the Garden of Eden and the Curtis–Hedlund–Lyndon theorems, as well as the balance property of surjective cellular automata. The domino problem is a classical undecidable decision problem whose variants are described in the chapter. Reductions from tiling problems to questions concerning cellular automata are also covered.

Chapter 9 by M. Hochman
Multidimensional shifts of finite type and sofic shifts

In Chapter 9 multidimensional shifts of finite type (SFTs) are examined from the language-theoretic and recursive-theoretic point of view, by specifically discussing the recent results of Hochman–Meyerovitch on the characterisation of entropies, Simpson's realisation theorem for degrees of computability, Hochman's characterization of 'slices' of SFTs (i.e., restrictions of their language to lower-dimensional lattices), the Jeandel–Vanier characterisation of sets of periods of SFTs; a variety of other related developments are also mentioned. A self-contained presentation of the basic definitions and results needed from symbolic dynamics and recursion theory is also included, but this chapter also relies on Robinson's aperiodic tile set, which is presented in Chapter 8. Modulo this, complete proofs for many of the main results above are given.

Chapter 10 by C. Reutenauer
Linearly recursive sequences and Dynkin diagrams

Following an idea of Caldero, in the realm of cluster algebras of Fomin and Zelevinsky, each acyclic quiver (i.e., a directed graph) defines a sequence of integers through a highly non-linear recursion. The Laurent phenomenon of Fomin and Zelevinsky implies that the number in the sequence are integers. It turns out that for certain quivers, the sequence satisfies, besides its non-linear defining recursion, also a linear recursion. The corresponding quivers are completely classified here: they are obtained by providing an acyclic orientation to a Dynkin graph, or an extended Dynkin graph. An important tool in order to give proofs is the concept of SL_2-tilings; this is a filling of the discrete plane by numbers in such a way that each connected two by two submatrix has determinant 1.

Chapter 11 by E. Cesaratto and B. Vallée
Pseudo-randomness of a random Kronecker sequence. An instance of dynamical analysis

In the last chapter of this volume, the focus is put on probabilistic features of the celebrated Kronecker sequence $\mathscr{K}(\alpha)$ formed of the fractional parts of the multiples of a real α, when α is randomly chosen in the unit interval. The authors are interested in measures of pseudo-randomness of the sequence, via five parameters (two distances, covered space, discrepancy and Arnold constant), and they then perform a probabilistic study of pseudo-randomness, in four various probabilistic settings. Indeed, the authors first deal with two 'unconstrained' probabilistic settings, where α is a random real, or α is a random rational. It is well-known that the behaviour of the sequence $\mathscr{K}(\alpha)$ heavily depends on the size of digits in the continued fraction expansion of α. This is why the authors also consider two 'constrained' probabilistic settings, where α is randomly chosen among the reals (or rationals) whose digits in the continued fraction expansion are bounded (by some M). The corresponding probabilistic studies are performed, exhibiting a great similarity between the rational and real settings, and the transition from the constrained model to the unconstrained model is studied when $M \to \infty$.

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The editors also thank Olivier Bodini and Thomas Fernique for the illustration on the cover of the book. This is a cropped view of a tiling by squares with two adjacent edges connected by a red wire (kind of 'half Truchet-tiles'). The presence/absence of wires is governed on the top row by the Thue–Morse sequence (from left to right) and on the left column by the Fibonacci sequence (from top to bottom). These two sequences completely determine the tiling on the whole bottom-right quarter of plane, which can thus be seen as a deterministic crisscross pattern of the Thue–Morse and Fibonacci sequences.