

Fourier Integrals in Classical Analysis

经典分析中的傅立叶积分

CHRISTOPHER D.SOGGE

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Fourier Integrals in Classical Analysis is an advanced monograph concerned with modern treatments of central problems in harmonic analysis. The main theme of the book is the interplay between ideas used to study the propagation of singularities for the wave equation and their counterparts in classical analysis. Using microlocal analysis, the author, in particular, studies problems involving maximal functions and Riesz means using the so-called half-wave operator.

This self-contained book starts with a rapid review of important topics in Fourier analysis. The author then presents the necessary tools from microlocal analysis, and goes on to give a proof of the sharp Weyl formula which he then modifies to give sharp estimates for the size of eigenfunctions on compact manifolds. Finally, at the end, the tools that have been developed are used to study the regularity properties of Fourier integral operators, culminating in the proof of local smoothing estimates and their applications to singular maximal theorems in two and more dimensions.

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To my family

Preface

Except for minor modifications, this monograph represents the lecture notes of a course I gave at UCLA during the winter and spring quarters of 1991. My purpose in the course was to present the necessary background material and to show how ideas from the theory of Fourier integral operators can be useful for studying basic topics in classical analysis, such as oscillatory integrals and maximal functions. The link between the theory of Fourier integral operators and classical analysis is of course not new, since one of the early goals of microlocal analysis was to provide variable coefficient versions of the Fourier transform. However, the primary goal of this subject was to develop tools for the study of partial differential equations and, to some extent, only recently have many classical analysts realized its utility in their subject. In these notes I attempted to stress the unity between these two subjects and only presented the material from microlocal analysis which would be needed for the later applications in Fourier analysis. I did not intend for this course to serve as an introduction to microlocal analysis. For this the reader should be referred to the excellent treatises of Hörmander [5], [7] and Treves [1].

In addition to these sources, I also borrowed heavily from Stein [4]. His work represents lecture notes based on a course which he gave at Princeton while I was his graduate student. As the reader can certainly tell, this course influenced me quite a bit and I am happy to acknowledge my indebtedness. My presentation of the overlapping material is very similar to his, except that I chose to present the material in the chapter on oscillatory integrals more geometrically, using the cotangent bundle. This turns out to be useful in dealing with Fourier analysis on manifolds and it also helps to motivate some results concerning Fourier integral operators, in particular the local smoothing estimates at the end of the monograph.

Roughly speaking, the material is organized as follows. The first two chapters present background material on Fourier analysis and stationary phase that will be used throughout. The next chapter deals with non-homogeneous oscillatory integrals. It contains the L^2 restriction theorem for the Fourier transform, estimates for Riesz means in \mathbb{R}^n , and Bourgain's circular maximal theorem. The goal of the rest of the monograph is mainly to develop generalizations of these results. The first step in this direction is to present some basic background material from the

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Chapter 0

Background

The purpose of this chapter and the next is to present the background material that will be needed. The topics are standard and a more thorough treatment can be found in many excellent sources, such as Stein [2] and Stein and Weiss [1] for the first half and Hörmander [7, Vol. 1] for the second.

We start out by rapidly going over basic results from real analysis, including standard theorems concerning the Fourier transform in \mathbb{R}^n and Calderón-Zygmund theory. We then apply this to prove the Hardy-Littlewood-Sobolev inequality. This theorem on fractional integration will be used throughout and we shall also present a simple argument showing how the n -dimensional theorem follows from the original one-dimensional inequality of Hardy and Littlewood. This type of argument will be used again and again. Finally, in the last two sections we give the definition of the wave front set of a distribution and compute the wave front sets of distributions which are given by oscillatory integrals. This will be our first encounter with the cotangent bundle and, as the monograph progresses, this will play an increasingly important role.

0.1. Fourier Transform

Given $f \in L^1(\mathbb{R}^n)$, we define its Fourier transform by setting

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle x, \xi \rangle} f(x) dx. \quad (0.1.1)$$

Given $h \in \mathbb{R}^n$, let $(\tau_h f)(x) = f(x + h)$. Notice that $\tau_{-h} e^{-i(\cdot, \xi)} = e^{i(h, \xi)} e^{-i(\cdot, \xi)}$ and so

$$(\tau_h f)^\wedge(\xi) = e^{i(h, \xi)} \hat{f}(\xi). \quad (0.1.2)$$

In a moment, we shall see that we can invert (0.1.1) (for appropriate f) and that we have the formula

$$f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x, \xi)} \hat{f}(\xi) d\xi. \quad (0.1.3)$$

Thus, the Fourier transform decomposes a function into a continuous sum of characters (eigenfunctions for translations).

Before turning to Fourier's inversion formula (0.1.3), let us record some elementary facts concerning the Fourier transform of L^1 functions.

Theorem 0.1.1:

- (1) $\|\hat{f}\|_\infty \leq \|f\|_1$.
- (2) If $f \in L^1$, then \hat{f} is uniformly continuous.

Theorem 0.1.2 (Riemann-Lebesgue): If $f \in L^1(\mathbb{R}^n)$, then $\hat{f}(\xi) \rightarrow 0$ as $\xi \rightarrow \infty$, and, hence, $\hat{f} \in C_0(\mathbb{R}^n)$.

Theorem 0.1.1 follows directly from the definition (0.1.1). To prove Theorem 0.1.2, one first notices from an explicit calculation that the result holds when f is the characteristic function of a cube. From this one derives Theorem 0.1.2 via a limiting argument.

Even though \hat{f} is in C_0 , the integral (0.1.3) will not converge for general $f \in L^1$. However, for a dense subspace we shall see that the integral converges absolutely and that (0.1.3) holds.

Definition 0.1.3: The set of Schwartz-class functions, $\mathcal{S}(\mathbb{R}^n)$, consists of all $\phi \in C^\infty(\mathbb{R}^n)$ satisfying

$$\sup_x |x^\gamma \partial^\alpha \phi(x)| < \infty, \quad (0.1.4)$$

for all multi-indices α, γ .¹

We give \mathcal{S} the topology arising from the semi-norms (0.1.4). This makes \mathcal{S} a Fréchet space. Notice that the set of all compactly supported C^∞ functions, $C_0^\infty(\mathbb{R}^n)$, is contained in \mathcal{S} .

Let $D_j = \frac{1}{i} \frac{\partial}{\partial x_j}$. Then we have:

Theorem 0.1.4: If $\phi \in \mathcal{S}$, then the Fourier transform of $D_j \phi$ is $\xi_j \hat{\phi}(\xi)$. Also, the Fourier transform of $x_j \phi$ is $-D_j \hat{\phi}$.

¹ Here $\alpha = (\alpha_1, \dots, \alpha_n)$, $\gamma = (\gamma_1, \dots, \gamma_n)$ and $x^\gamma = x^{\gamma_1} \dots x^{\gamma_n}$, $\partial^\alpha = (\partial/\partial x_1)^{\alpha_1} \dots (\partial/\partial x_n)^{\alpha_n}$.

Proof: To prove the second assertion we differentiate (0.1.1) to obtain

$$D_j \hat{\phi}(\xi) = \int e^{-i\langle x, \xi \rangle} (-x_j) \phi(x) dx,$$

since the integral converges uniformly. If we integrate by parts, we see that

$$\xi_j \hat{\phi}(\xi) = \int -D_j e^{-i\langle x, \xi \rangle} \cdot \phi(x) dx = \int e^{-i\langle x, \xi \rangle} D_j \phi(x) dx,$$

which is the first assertion. ■

Notice that Theorem 0.1.4 implies the formula

$$\xi^\alpha D^\gamma \hat{\phi}(\xi) = \int e^{-i\langle x, \xi \rangle} D^\alpha ((-x)^\gamma \phi(x)) dx. \quad (0.1.5)$$

If we set $C = \int (1 + |x|)^{-n-1} dx$, then this leads to the estimate

$$\sup_{\xi} |\xi^\gamma D^\alpha \hat{\phi}(\xi)| \leq C \sup_x (1 + |x|)^{n+1} |D^\gamma (x^\alpha \phi(x))|. \quad (0.1.6)$$

Inequality (0.1.6) of course implies that the Fourier transform maps \mathcal{S} into itself. However, much more is true:

Theorem 0.1.5: *The Fourier transform $\phi \rightarrow \hat{\phi}$ is an isomorphism of \mathcal{S} into \mathcal{S} whose inverse is given by Fourier's inversion formula (0.1.3).*

The proof is based on a couple of lemmas. The first is the multiplication formula for the Fourier transform:

Lemma 0.1.6: *If $f, g \in L^1$ then*

$$\int_{\mathbb{R}^n} \hat{f} g dx = \int_{\mathbb{R}^n} f \hat{g} dx.$$

The next is a formula for the Fourier transform of Gaussians:

Lemma 0.1.7: $\int_{\mathbb{R}^n} e^{-i\langle x, \xi \rangle} e^{-\varepsilon|x|^2/2} dx = (2\pi/\varepsilon)^{n/2} e^{-|\xi|^2/2\varepsilon}.$

The first lemma is easy to prove. If we apply (0.1.1) and Fubini's theorem, we see that the left side equals

$$\begin{aligned} \int \left\{ \int f(y) e^{-i\langle x, y \rangle} dy \right\} g(x) dx &= \int \left\{ \int e^{-i\langle x, y \rangle} g(x) dx \right\} f(y) dy \\ &= \int \hat{g} f dy. \end{aligned}$$

It is also clear that Lemma 0.1.7 must follow from the special case where $n = 1$. But

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-t^2/2} e^{-it\tau} dt &= e^{-\tau^2/2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(t+i\tau)^2} dt \\ &= e^{-\tau^2/2} \int_{-\infty}^{\infty} e^{-t^2/2} dt \\ &= \sqrt{2\pi} e^{-\tau^2/2}.\end{aligned}$$

In the second step we have used Cauchy's theorem. If we make the change of variables $\varepsilon^{1/2}s = t$ in the last integral, we get the desired result.

Proof of Theorem 0.1.5: We must prove that when $\phi \in \mathcal{S}$,

$$\phi(x) = (2\pi)^{-n} \int e^{i\langle x, \xi \rangle} \hat{\phi}(\xi) d\xi.$$

By the dominated convergence theorem, the right side equals

$$\lim_{\varepsilon \rightarrow 0} (2\pi)^{-n} \int e^{i\langle x, \xi \rangle} \hat{\phi}(\xi) e^{-\varepsilon|\xi|^2/2} d\xi.$$

If we recall (0.1.2), then we see that this equals

$$\lim_{\varepsilon \rightarrow 0} (2\pi\varepsilon)^{-n/2} \int \phi(x+y) e^{-|y|^2/2\varepsilon} dy.$$

Finally, since $(2\pi)^{-n/2} \int e^{-|y|^2/2} dy = 1$, it is easy to check that the last limit is $\phi(x)$. ■

If for $f, g \in L^1$ we define convolution by

$$(f * g)(x) = \int f(x-y)g(y) dy,$$

then another fundamental result is:

Theorem 0.1.8: If $\phi, \psi \in \mathcal{S}$ then

$$(2\pi)^n \int \phi \bar{\psi} dx = \int \hat{\phi} \bar{\hat{\psi}} d\xi, \quad (0.1.7)$$

$$(\phi * \psi)^\wedge(\xi) = \hat{\phi}(\xi) \hat{\psi}(\xi); \quad (0.1.8)$$

$$(\phi\psi)^\wedge(\xi) = (2\pi)^{-n} (\hat{\phi} * \hat{\psi})(\xi). \quad (0.1.9)$$

To prove (0.1.7), set $\chi = (2\pi)^{-n} \bar{\hat{\psi}}$. Then the Fourier inversion formula implies that $\hat{\chi} = \bar{\psi}$. Consequently, (0.1.7) follows from Lemma 0.1.6. We leave the other two formulas as exercises.