BIRKHÄUSER

Modern Birkhäuser Classics

Discriminants, Resultants, and Multidimensional Determinants

I. M. Gelfand
M. M. Kapranov
A. V. Zelevinsky

判别式、结式和多维行列式

Birkhäuser

世界图书出版公司
www.wpcbj.com.cn

Discriminants, Resultants and Multidimensional Determinants

I.M. Gelfand
M.M. Kapranov
A.V. Zelevinsky

常州大学山书馆
藏书章

Reprint of the 1994 Edition

Birkhäuser
Boston • Basel • Berlin

Israel M. Gelfand
Department of Mathematics
Rutgers University
New Brunswick, NJ 08903
U.S.A.

Mikhail M. Kapranov
Department of Mathematics
Yale University
New Haven, CT 06520-8283
U.S.A.

Andrei V. Zelevinsky
Department of Mathematics
Northeastern University
Boston, MA 02115
U.S.A.

Originally published in the series *Mathematics: Theory and Applications*

ISBN-13: 978-0-8176-4770-4 e-ISBN-13: 978-0-8176-4771-1
DOI: 10.1007/978-0-8176-4771-1

Library of Congress Control Number: 2007940668

Mathematics Subject Classification (2000): 14M25, 14M12, 13D25, 13XX, 14XX, 13C40, 15A15, 13F20, 52B20

©2008 Birkhäuser Boston
All rights reserved. This work may not be translated or copied in whole or in part without the written permission of the publisher (Birkhäuser Boston, c/o Springer Science+Business Media LLC, 233 Spring Street, New York, NY 10013, USA), except for brief excerpts in connection with reviews or scholarly analysis. Use in connection with any form of information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed is forbidden.
The use in this publication of trade names, trademarks, service marks and similar terms, even if they are not identified as such, is not to be taken as an expression of opinion as to whether or not they are subject to proprietary rights.

Reprint from English language edition:
Discriminants, Resultants, and Multidimensional Determinants
by I. M. Gelfand, M.M.Kapranov, A. V.Zelevinsky
Copyright © 2008, Birkhäuser Boston
Birkhäuser Boston is a part of Springer Science+Business Media
All Rights Reserved

This reprint has been authorized by Springer Science & Business Media for distribution in China Mainland only and not for export therefrom.

图书在版编目（CIP）数据

判别式结式和多维行列式 = Discriminants，Resultants，and Multidimensional Determinants：英文/（美）盖尔芬德（Gelfand，I. M.）著. — 影印本. —北京：世界图书出版公司北京公司，2013. 5
ISBN 978 - 7 - 5100 - 6145 - 5

Ⅰ. ①判… Ⅱ. ①盖… Ⅲ. ①结式—英文②行列式—英文 Ⅳ. ①O151. 1 ②O151. 22

中国版本图书馆 CIP 数据核字（2013）第 088359 号

书　　名：Discriminants，Resultants，and Multidimensional Determinants
作　　者：I. M. Gelfand，M. M. Kapranov，A. V. Zelevinsky
中 译 名：判别式、结式和多维行列式
责任编辑：高蓉　刘慧

出 版 者：世界图书出版公司北京公司
印 刷 者：三河市国英印务有限公司
发　　行：世界图书出版公司北京公司（北京朝内大街 137 号 100010）
联系电话：010 - 64021602，010 - 64015659
电子信箱：kjb@ wpcbj. com. cn

开　　本：24 开
印　　张：22. 5
版　　次：2013 年 10 月
版权登记：图字：01 - 2013 - 8538

书　　号：978 - 7 - 5100 - 6145 - 5　　　定　　价：79. 00 元

Job #: 111365

Author Name: Gelfand

Title of Book: Discriminants, Resultants and Multidimen

ISBN #: 9780817647704

此为试读,需要完整PDF请访问: www.ertongbook.com

Modern Birkhäuser Classics

Many of the original research and survey monographs in pure and applied mathematics published by Birkhäuser in recent decades have been groundbreaking and have come to be regarded as foundational to the subject. Through the MBC Series, a select number of these modern classics, entirely uncorrected, are being re-released in paperback (and as eBooks) to ensure that these treasures remain accessible to new generations of students, scholars, and researchers.

Preface

This book has expanded from our attempt to construct a general theory of hypergeometric functions and can be regarded as a first step towards its systematic exposition. However, this step turned out to be so interesting and important, and the whole program so overwhelming, that we decided to present it as a separate work. Moreover, in the process of writing we discovered a beautiful area which had been nearly forgotten so that our work can be regarded as a natural continuation of the classical developments in algebra during the 19th century.

We found that Cayley and other mathematicians of the period understood many of the concepts which today are commonly thought of as modern and quite recent. Thus, in an 1848 note on the resultant, Cayley in fact laid out the foundations of modern homological algebra. We were happy to enter into spiritual contact with this great mathematician.

The place of discriminants in the general theory of hypergeometric functions is similar to the place of quasi-classical approximation in quantum mechanics. More precisely, in [GGZ] [GKZ2] [GZK1] a general class of special functions was introduced and studied, the so-called A-hypergeometric functions. These functions satisfy a certain holonomic system of linear partial differential equations (the A-hypergeometric equations). The A-discriminant, which is one of our main objects of study, describes singularities of A-hypergeometric functions. According to the general principles of the theory of linear differential equations, these singularities are governed by the vanishing of the highest symbols of A-hypergeometric equations. The relation between differential operators and their highest symbols is the mathematical counterpart of the relation between quantum and classical mechanics; so we can say that hypergeometric functions provide a "quantization" of discriminants.

In our work on hypergeometric functions we found connections with many questions in algebra and combinatorics. We hope that this book brings to light some of these connections. One of the algebraic concepts which seems to us particularly important is that of hyperdeterminants (analogs of determinants for multi-dimensional "matrices.") After rediscovering hyperdeterminants in connection with hypergeometric functions, we found that they too, had been introduced by Cayley in the 1840s. Unfortunately, later on, the study of hyperdeterminants was largely abandoned in favor of another, more straightforward definition (cf. [P]). The only other work on hyperdeterminants of which we are aware is an important paper by Schläfli [Schl]. In this volume we give a detailed treatment of hyperdeterminants with the hope of attracting the attention of other mathematicians to this subject.

We would like to thank S.I. Gelfand, M.I. Graev and V.A. Vassiliev, who, through discussions and collaboration, have much influenced our understanding of the vast and beautiful field of hypergeometric functions.

Contents

I. GENERAL DISCRIMINANTS AND RESULTANTS

CHAPTER 1. Projective Dual Varieties and General Discriminants

CHAPTER 2. The Cayley Method for Studying Discriminants

CHAPTER 3. Associated Varieties and General Resultants

CHAPTER 4. Chow Varieties

此为试读,需要完整PDF请访问: www.ertongbook.com

II. A-DISCRIMINANTS AND A-RESULTANTS

CHAPTER 5. Toric Varieties

CHAPTER 6. Newton Polytopes and Chow Polytopes

CHAPTER 7. Triangulations and Secondary Polytopes

CHAPTER 8. A-Resultants and Chow Polytopes of Toric Varieties

CHAPTER 9. A-Discriminants

CHAPTER 10. Principal A-Determinants

Introduction

I

In this book we study discriminants and resultants of polynomials in several variables. The most familiar example is the discriminant of a quadratic polynomial $f(x) = ax^2 + bx + c$. This is

$$\Delta(f) = b^2 - 4ac, \tag{1}$$

which vanishes when $f(x)$ has a double root.

More generally, we can consider a polynomial $f(x_1, \ldots, x_k)$ of degree $\leq d$ in k variables. An analog of a multiple root for f is a point where f vanishes together with all its first partial derivatives $\partial f/\partial x_i$. The *discriminant* $\Delta(f)$ is a polynomial function in the coefficients of f which vanishes whenever f has such a "multiple root." The existence of Δ is not quite trivial; however, it can be shown that $\Delta(f)$ exists and is unique up to sign if we require it to be irreducible and to have relatively prime integer coefficients. For instance, the discriminant of a cubic polynomial in one variable ($k = 1$, $d = 3$) is given by

$$\Delta(a_0+a_1x+a_2x^2+a_3x^3) = a_1^2a_2^2-4a_1^3a_3-4a_0a_2^3-27a_0^2a_3^2+18a_0a_1a_2a_3. \tag{2}$$

There is a subtle point in the definition of $\Delta(f)$: that is, $\Delta(f)$ depends not only on f but also on the choice of a degree bound d. For instance, the formula (2) applied to a quadratic polynomial gives a different expression from (1). With this in mind, we introduce the following more general version of a discriminant. Let A be a finite set of monomials in k variables, and let $\mathbf{C}^A$ denote the space of all polynomials with complex coefficients all of whose monomials belong to A. The *A-discriminant* $\Delta_A(f)$ is an irreducible polynomial in the coefficients of $f \in \mathbf{C}^A$ which vanishes whenever f has a multiple root $(x_1, \ldots, x_k)$ with all $x_i \neq 0$ (the last condition is added to be able to ignore trivial multiple roots which can appear if all monomials from A have high degree). The A-discriminant will be one of our main objects of study.

The notion of the A-discriminant includes as special cases several fundamental algebraic concepts. If we take $A = \{1, x, \ldots, x^m, y, yx, \ldots, yx^n\}$, for example, then a typical polynomial from $\mathbf{C}^A$ has the form $f(x) + yg(x)$. Its A-discriminant is the *resultant* of f and g: it vanishes whenever f and g have a common root.

More generally, the resultant of $k + 1$ polynomials $f_0, \ldots, f_k$ in k variables is defined as an irreducible polynomial in the coefficients of $f_0, \ldots, f_k$, which vanishes whenever these polynomials have a common root. The resultant can be treated as a special case of the A-discriminant of an auxiliary polynomial $f_0(x) + \sum_{i=1}^{k} y_i f_i(x)$, $x = (x_1, \ldots, x_k)$.

Another important example occurs when A consists of n^2 monomials $x_i y_j$, $i, j = 1, \ldots, n$. A typical polynomial from $\mathbf{C}^A$ is now a bilinear form $f(x, y) = \sum a_{ij} x_i y_j$ whose A-discriminant is the determinant of the matrix $\|a_{ij}\|$.

The last example has a natural generalization: we can take A as the set of all multilinear monomials in three or more groups of variables. An element $f \in \mathbf{C}^A$ (i.e., a multilinear form) is represented by a higher-dimensional "matrix" $\|a_{i_1 \ldots i_r}\|$. Thus the A-discriminant Δ_A in this case is a polynomial function of a "matrix" which extends the notion of a determinant. Following Cayley [Ca1], we call this Δ_A the *hyperdeterminant* of $\|a_{i_1 \ldots i_r}\|$. For example, the hyperdeterminant of a $2 \times 2 \times 2$ matrix $\|a_{ijk}\|$, $i, j, k = 0, 1$, is given by

$$(a_{000}^2 a_{111}^2 + a_{001}^2 a_{110}^2 + a_{010}^2 a_{101}^2 + a_{011}^2 a_{100}^2)$$

$$-2(a_{000}a_{001}a_{110}a_{111} + a_{000}a_{010}a_{101}a_{111} + a_{000}a_{011}a_{100}a_{111} + a_{001}a_{010}a_{101}a_{110}$$

$$+a_{001}a_{011}a_{110}a_{100} + a_{010}a_{011}a_{101}a_{100}) + 4(a_{000}a_{011}a_{101}a_{110} + a_{001}a_{010}a_{100}a_{111}).$$

The study of hyperdeterminants was initiated by Cayley [Ca1] and Schläfli [Schl] but then was largely abandoned for 150 years. We present a treatment of hyperdeterminants in Chapter 14.

II

Let $\nabla_A = \{f \in \mathbf{C}^A : \Delta_A(f) = 0\}$ be the hypersurface in the space of polynomials consisting of polynomials with vanishing A-discriminant. We shall be mainly concerned with the following two closely related problems:

(a) the study of the geometric properties of the hypersurface ∇_A;

(b) finding an explicit algebraic expression of the discriminant Δ_A.

To illustrate the importance of problem (a), consider the special case when A consists of all monomials in $x_1, \ldots, x_k$ of a given degree d. Every $f \in \mathbf{C}^A$ (i.e., a homogeneous form of degree d) defines a hypersurface $\{f = 0\}$ in the projective space P^{k-1}. It is easy to see that ∇_A consists exactly of those f for which the hypersurface $\{f = 0\}$ is *singular*. Therefore the complement $\mathbf{C}^A - \nabla_A$ parametrizes all smooth hypersurfaces of a given degree in the projective space. To understand the geometric structure of $\mathbf{C}^A - \nabla_A$ is an important instance of the general moduli problem in algebraic geometry.

Equally important is the situation over the real numbers. Hilbert's 16th problem (classifying isotopy types of smooth real hypersurfaces of given degree d)

amounts to the study of connected components of $\mathbf{R}^A - \nabla_A$, the space of real polynomials with a non-vanishing discriminant.

Problem (b) has a long and glorious history. Explicit formulas for discriminants and resultants were the focus of several remarkable mathematicians in the last century. Many ingenious formulas were found by Cayley, Sylvester and their followers. However, we are still very far from a complete understanding of discriminants. For instance, an explicit polynomial expression for Δ_A is known only in a very limited number of special cases. Such formulas would be of great importance for the problem of finding explicit solutions of systems of polynomial equations. Problems of this kind are of interest not only for theoretical reasons, but are encountered more and more on a practical level because of the progress in computer technology.

III

We will use three main approaches in our study of discriminants and resultants:

- a geometric approach via projective duality and associated hypersurfaces;
- an algebraic approach via homological algebra and determinants of complexes (Whitehead torsion);
- a combinatorial approach via Newton polytopes and triangulations.

The geometric approach to discriminants is based on the observation that the discriminantal variety ∇_A is projectively dual to a certain variety X_A defined by a simple parametric representation. For example, if A consists of all monomials of degree d in k variables then X_A is the projective space P^{k-1} in its Veronese embedding. In the general case, X_A is the projective *toric* variety associated with A. The notion of the projectively dual variety $X^\vee$ makes sense for an arbitrary projective variety $X \subset P^{n-1}$: it is the closure of the set of all hyperplanes in P^{n-1} which are tangent to X at some smooth point. Thus the problem of finding the discriminant is a particular case of a more general geometric problem: find the equation(s) of $X^\vee$. We call this equation (in the case where $X^\vee$ is a hypersurface) the *X-discriminant*.

Although the resultants can be formally treated as discriminants of a special kind (see above), they have their own interesting geometric meaning. As for discriminants, we can associate the resultant to any projective variety $X \subset P^{n-1}$. Instead of $X^\vee$, we now consider the *associated hypersurface* $\mathcal{Z}(X)$ of X. If $\dim X = k-1$ then $\mathcal{Z}(X)$ is the locus of all codimension k projective subspaces in P^{n-1} which meet X. The equation of $\mathcal{Z}(X)$ in the appropriate Grassmannian is the classical *Chow form* of X. This can be represented as a polynomial in the

此为试读，需要完整PDF请访问：www.ertongbook.com